

More Logarithmic-Factor Speedups for 3SUM, (median,+)-Convolution, and Some Geometric 3SUM-Hard Problems

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November 1, 2017

Abstract

We present an algorithm that solves the 3SUM problem for n real numbers in $O((n^2/\log^2 n)(\log \log n)^{O(1)})$ time, improving previous solutions by about a logarithmic factor. Our framework for shaving off two logarithmic factors can be applied to other problems, such as (median,+)-convolution/matrix multiplication and algebraic generalizations of 3SUM. We also obtain the first subquadratic results on some 3SUM-hard problems in computational geometry, for example, deciding whether (the interiors of) a constant number of simple polygons have a common intersection.

1 Introduction

3SUM. The starting point of this paper is the *3SUM* problem:

Given sets A , B , and C of n real numbers, decide whether there exists a triple $(a, b, c) \in A \times B \times C$ with $c = a + b$.¹

The problem has received considerable attention by algorithm researchers, and understanding the complexity of the problem is fundamental to the field. The conjecture that it cannot be solved in $O(n^{2-\epsilon})$ time (in the real or integer setting) has been used as a basis for proving conditional lower bounds for numerous problems from a variety of areas (computational geometry, data structures, string algorithms, and so on). See previous papers (such as [22]) for more background.

In a surprising breakthrough, Grønlund and Pettie [22] discovered the first subquadratic algorithms for 3SUM. They showed the decision-tree complexity of the problem is $O(n^{3/2}\sqrt{\log n})$, and gave a randomized $O((n^2/\log n)(\log \log n)^2)$ -time algorithm and a deterministic $O((n^2/\log^{2/3} n)(\log \log n)^{2/3})$ -time algorithm in the standard real-RAM model. Small improvements were subsequently reported by Freund [19] and Gold and Sharir [21], who independently found $O((n^2/\log n) \log \log n)$ -time deterministic algorithms (the latter also eliminated the $\sqrt{\log n}$ factor from the decision-tree complexity bound). In another dramatic breakthrough, Kane, Lovett, and Moran [24] developed a technique for obtaining near-optimal decision-tree complexity for many problems, and in particular, a near-linear $O(n \log^2 n)$ decision-tree upper bound for 3SUM; their

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¹The original formulation seeks a triple with $a + b + c = 0$, which is equivalent after negating the elements of C .

technique does not seem to have new implications on the (uniform) time complexity of 3SUM (although it probably could lead to yet another $O((n^2/\log n)(\log \log n)^{O(1)})$ -time algorithm).

In this paper, we give a further improvement on the time complexity, by about one more logarithmic factor: we present a new deterministic $O((n^2/\log^2 n)(\log \log n)^{O(1)})$ -time algorithm.

Ignoring $\log \log n$ factors, this matches known results for 3SUM in the special case of integer input, where a randomized $O((n^2/\log^2 n)(\log \log n)^2)$ time bound can be obtained via hashing techniques [5]. In contrast, besides being more general, our algorithm is deterministic and can also solve variations of the problem, e.g., finding the closest number (predecessor or successor) in $\{a + b : (a, b) \in A \times B\}$ to each $c \in C$.

The development on 3SUM parallels the history of combinatorial algorithms for the *all-pairs shortest paths (APSP)* problem for dense real-edge-weighted graphs, or equivalently, the *(min, +)-matrix multiplication* problem for real matrices. In fact, the slightly subquadratic algorithms by Grønlund and Pettie and the subsequent refinements by Freund or Gold and Sharir all used a geometric subproblem, *dominance searching* in logarithmic dimensions, following the author’s $O(n^3/\log n)$ -time APSP algorithm [10]. A later paper by the author [11] gave a further improved $O((n^3/\log^2 n)(\log \log n)^{O(1)})$ -time algorithm for APSP, using a different geometric approach, via *cuttings* in near-logarithmic dimensions. Our improvement will follow the approach in [11]. The analogy between these 3SUM and APSP algorithms makes sense in hindsight, but isn’t immediately obvious, at least to this author (which explains why this paper wasn’t written right after Grønlund and Pettie’s breakthrough), and there are some new technical challenges (for example, involving bit packing and, at some point, simulation of sorting networks).

(select, +)-Convolution/Matrix Multiplication. The way Grønlund and Pettie and subsequent authors used dominance to solve 3SUM actually more resembled a previous algorithm by Bremner et al. [7] on another related problem, *(median, +)-convolution*, or more generally, what we will call *(select, +)-convolution*:

Given real sequences $A = \langle a_1, \dots, a_n \rangle$ and $B = \langle b_1, \dots, b_n \rangle$ and integers k_1, \dots, k_n , compute $c_i =$ the k_i -th smallest of $\{a_u + b_{i-u} : u \in \{1, \dots, i-1\}\}$, for every $i \in \{1, \dots, n\}$.

(This problem was used to solve the “necklace alignment problem” under ℓ_1 distances [7].) Our ideas can also improve Bremner et al.’s $O((n^2/\log n)(\log \log n)^{O(1)})$ time bound to $O((n^2/\log^2 n)(\log \log n)^{O(1)})$ for that problem.

We can similarly solve the *(select, +)-matrix multiplication* problem in $O((n^3/\log^2 n)(\log \log n)^{O(1)})$ time (we are not aware of any prior work on this problem):

Given two real $n \times n$ matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ and an integer matrix $K = \{k_{ij}\}$, compute $c_{ij} =$ the k_{ij} -th smallest of $\{a_{iu} + b_{uj} : u \in \{1, \dots, n\}\}$, for every $i, j \in \{1, \dots, n\}$.

Algebraic 3SUM. Barba et al. [6] recently gave an $O((n^2/\sqrt{\log n})(\log \log n)^{O(1)})$ -time algorithm for a generalization of 3SUM, which we will refer to as *algebraic 3SUM*:

Let $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function of *constant description complexity*, i.e., $\{(x, y, z) \in \mathbb{R}^3 : \varphi(x, y) = z\}$ is a semi-algebraic set of constant degree. Given sets A, B , and C of n real numbers, decide whether there exists a triple $(a, b, c) \in A \times B \times C$ with $\varphi(a, b) = c$.

Our ideas can also lead to a faster $O((n^2/\log^2 n)(\log \log n)^{O(1)})$ -time algorithm for this problem.

Geometric 3SUM-Hard Problems. The importance of 3SUM stems from the many problems that were shown to be 3SUM-hard, starting with the seminal paper by Gajentaan and Overmars [20] in computational geometry. Grønlund and Pettie’s breakthrough was exciting because it gave hope to the possibility that slightly subquadratic algorithms might exist for these geometric problems as well. However, thus far, no such geometric result has materialized. (In contrast, examples of “*orthogonal-vectors-hard*” problems in computational geometry with slightly subquadratic algorithms was known earlier [1, 8, 9, 21].) Barba et al.’s work came close: their algebraic 3SUM algorithm can be used to solve a standard geometric 3SUM-hard problem—testing whether a given set of n points is degenerate, i.e., contain a collinear triple—but only in the special case when the points lie on a small number of fixed-degree algebraic curves in 2D. The input there is still intrinsically one-dimensional.

In this paper, we show how to adapt our techniques to obtain an $O((n^2/\log^2 n)(\log \log n)^{O(1)})$ -time algorithm for some genuinely two-dimensional problems:

- (i) *Intersection of 3 polygons.* Given 3 simple polygons with n vertices, decide whether the common intersection is empty.²
- (ii) *Coverage by 3 polygons.* Given 3 simple polygons with n vertices and a triangle Δ_0 , decide whether the union of the polygons covers Δ_0 .
- (iii) *Degeneracy testing for $O(1)$ -chromatic line segments.* Given $O(1)$ sets each consisting of n *disjoint* line segments in \mathbb{R}^2 , decide whether there exist 3 line segments meeting at a common point.
- (iv) *Offline triangle range searching for bichromatic segment intersections.* Given sets A and B each consisting of n *disjoint* line segments, and a set C of n triangles in \mathbb{R}^2 , count the number of intersection points between A and B that are inside each triangle $c \in C$.³

Furthermore, we obtain a slightly slower $O((n^2/\log n)(\log \log n)^{O(1)})$ -time algorithm for the following related problems:

- (v) *Intersection of $O(1)$ polygons.* Given $O(1)$ simple polygons with n vertices, decide whether the common intersection is empty.
- (vi) *Coverage by $O(1)$ polygons.* Given $O(1)$ simple polygons with n vertices and a triangle Δ_0 , decide whether the union of the polygons covers Δ_0 .
- (vii) *Offline reverse triangle range searching for bichromatic segment intersections.* Given sets A and B each consisting of n *disjoint* line segments, and a set C of n triangles in \mathbb{R}^2 , count the number of triangles in C containing each intersection point q between A and B . Here, we may use any implicit output representation that allows us to retrieve the count for any given q , and the minimum/maximum count, in constant time.

It is not difficult to see that all the above problems are 3SUM-hard. Moreover, (i)–(iii) reduce to (iv), and (v)–(vi) reduce to (vii): (i) reduces to (iv), since if the intersection of the 3 polygons is

²This problem was observed to be 3SUM-hard in an unpublished note http://tcs.postech.ac.kr/workshops/kwgc05/problems/smallest_circle.pdf (2005).

³An online query version of this problem was recently considered by de Berg et al. [16], who also briefly noted connection to 3SUM.

nonempty, we can triangulate the polygons, and some vertex or some intersection point between 2 edges of 2 polygons would be inside a triangle of the other polygon; (ii) reduces to (i) by taking complements; (iii) reduces to (iv) by examining each triple of sets and viewing line segments as degenerate triangles; (v) reduces to (vii), since if the intersection of the polygons is nonempty, we can triangulate the complements of polygons, and some vertex or some intersection between 2 edges of 2 polygons would lie in no triangles of the complements of the other polygons; (vi) reduces to (v) by taking complements. The results for (i), (ii), (v), and (vi) hold even if each polygon has holes (or is a disconnected collection of disjoint polygons). The time bound remains subquadratic for a nonconstant number of polygons, if the number is less than $\log^\delta n$ for some δ .

To summarize, our main contributions are:

- a general recipe on how to shave off (in most cases) two logarithmic factors for a host of problems—for “Four-Russians”-style algorithms, two logarithmic factors are usually the most one could eliminate (see Section 8 for exceptions, which tend to be more limited in their applicability).
- identification of the first natural 3SUM-hard problems in computational geometry with slightly subquadratic algorithms—this is perhaps the most important “qualitative” contribution of the paper, to those who care less about the precise number of logarithmic factors shaved.

2 Preliminaries

Let $[n]$ denote $\{1, 2, \dots, n\}$. For a list A , let $A[i]$ denote its i -th element.

We assume a real-RAM model, where a word can hold an input real number or a w -bit number/pointer for a fixed $w = \Omega(\log n)$. The only operations on reals are evaluating the signs of constant-degree polynomials over a constant number of input values (in fact, for our 3SUM algorithms, all comparisons on reals will be of the form $a + b \leq c$ or $a + b \leq a' + b'$). For convenience, we assume that the model supports some nonstandard operations on w -bit words in constant time. By setting $w = \delta_0 \log n$ for a sufficiently small constant $\delta_0 > 0$, nonstandard word operations can be simulated by table lookup after an initial preprocessing of $2^{O(w)} = n^{O(\delta_0)}$ time.

We mention some simple facts about bit packing which will be of use later:

Fact 2.1. *Let $\langle x_1, \dots, x_n \rangle$ be a sequence of n numbers in $[b]$ stored in $O((n \log b)/w + 1)$ words.*

- We can sort the sequence in $O((n \log^2 b)/w + 1)$ time.*
- Given a table $f : [b] \rightarrow [b]$, we can compute $\langle f[x_1], \dots, f[x_n] \rangle$ in $O((n \log^2(bw))/w + b)$ time.*

Proof. (Review) Part (a) is well known [4] and can be obtained by a variant of mergesort. For (b), a similar time bound of $O((n \log^2(bn))/w + b)$ can be found in [11, Lemma 2.3(d)]: roughly, we sort the list of pairs (i, x_i) by x_i using (a), split the list into sublists with a common x_i , replace x_i with $f[x_i]$ in each sublist, concatenate these sublists, and finally sort all pairs by i to get back the original order. To replace the $\log(bn)$ factors by $\log(bw)$, divide the input sequence into blocks of $n_0 := bw/\log^2(bw)$ elements each, and apply the procedure to each block, giving total time $O((n/n_0) \cdot ((n_0 \log^2(bn_0))/w + b)) = O((n \log^2(bw))/w)$ for $n \geq n_0$. \square

3 3SUM

For two lists A and B , let $A + B$ denote the multiset $\{a + b : a \in A, b \in B\}$.

3.1 Reduction to Batched $A + B$ Searching/Selection

We formulate two subproblems, which will be the key to solving 3SUM:

Problem 3.1. (Batched $A + B$ Searching) *Given lists (“groups”) $A_1, \dots, A_m, B_1, \dots, B_m$ of d real numbers each, and given “query” lists C_{ij} of values in \mathbb{R} with $\sum_{i,j} |C_{ij}| = Q$, find the predecessor of c in $A_i + B_j$ for each $c \in C_{ij}$ and each $i, j \in [m]$.*

Problem 3.2. (Batched $A + B$ Selection) *Given lists (“groups”) $A_1, \dots, A_m, B_1, \dots, B_m$ of d real numbers each, and given “query” lists K_{ij} of numbers in $[d^2]$ with $\sum_{i,j} |K_{ij}| = Q$, find the k -th smallest in $A_i + B_j$ for each $k \in K_{ij}$ and each $i, j \in [m]$.*

Let $T_{\text{search},+}(m, d, Q)$ and $T_{\text{select},+}(m, d, Q)$ be the time complexity of these two problems. Note that $T_{\text{search},+}(m, d, Q) = O((T_{\text{select},+}(m, d, Q) + Q + m^2) \cdot \log d)$, since Problem 3.1 reduces to $O(\log(d^2))$ instances of Problem 3.2, by simultaneous binary searches for the rank of c for all $c \in C_{ij}$ and all i, j .

3SUM clearly reduces to Problem 3.1 with $m = 1$ and $d = Q = n$. The following simple lemma, based on Grønlund and Pettie’s grouping approach [22], shows a better reduction, to instances with smaller group size d :

Lemma 3.3. *3SUM can be solved in time $O(T_{\text{search},+}(n/d, d, n^2/d) + n^2/d + n \log n)$ for any given $d \leq n$.*

Proof. Sort A and B . Divide A into sublists $A_1, \dots, A_{n/d}$ of size d , and B into sublists $B_1, \dots, B_{n/d}$ of size d . For each $c \in C$, put c in C_{ij} iff $c \in [A_i[1] + B_j[1], A_{i+1}[1] + B_{j+1}[1])$. For each c , the (i, j) pairs satisfying this condition form a monotone sequence in $[n/d]^2$ (nondecreasing in the first coordinate and nonincreasing in the second); thus, there are $O(n/d)$ such pairs and they can be found by a linear scan over the two lists $\langle A_1[1], \dots, A_{n/d}[1] \rangle$ and $\langle B_1[1], \dots, B_{n/d}[1] \rangle$ in $O(n/d)$ time. So, $\sum_{i,j} |C_{ij}| = O(n^2/d)$, and the total time to generate all C_{ij} ’s is $O(n^2/d)$. For each $c \in C$, the predecessor of c in $A + B$ is the maximum among the predecessors of c in $A_i + B_j$ for all (i, j) with $c \in C_{ij}$. This gives an instance of Problem 3.1 with $O(n/d)$ sets of size d and total query list size $Q = O(n^2/d)$. \square

3.2 Batched $A + B$ Selection via Cuttings in Near-Logarithmic Dimensions

We now solve the batched $A + B$ selection problem for group size d near $\log m$, using a geometric approach based on the author’s APSP algorithm [11]. We need one tool from computational geometry:

Fact 3.4. (Cutting Lemma) *Given $r \leq n$ and a set H of n hyperplanes in \mathbb{R}^d , we can cut \mathbb{R}^d into $d^{O(d)} r^{O(d)}$ disjoint cells so that the relative interior of each cell intersects at most n/r hyperplanes of H .*

Furthermore, given a set P of n points, for every cell Δ with $P \cap \Delta \neq \emptyset$, we can generate the set $P_\Delta = P \cap \Delta$ and a set H_Δ of size at most n/r containing all hyperplanes of H intersecting the relative interior of Δ , in $d^{O(d)} nr^{O(d)}$ total time.

Proof. (Review) If randomization is allowed, a simple algorithm is to draw a random sample of about r hyperplanes and take the cells from a *canonical triangulation* of the *arrangement* of these hyperplanes. Derandomization is more involved. See [15] for a survey.

In our application, the hyperplanes are orthogonal to only a small number ($d^{O(1)}$) of directions. In this case, there is a much easier proof, as noted in [11]: roughly, select r hyperplanes (r quantiles) for each direction, and take the cells from the arrangement of the resulting $d^{O(1)}r$ hyperplanes. \square

We begin with one solution to batched $A + B$ selection in Theorem 3.5 below. It will be superseded by the later version of Theorem 3.6, but serves as a good warm-up of the basic idea.

Theorem 3.5. (Warm-Up Version) $T_{\text{select},+}(m, d, Q) = O(Q + m^2)$ for $d \leq \delta \log m / \log \log m$ for a sufficiently small constant $\delta > 0$.

Proof. For each A_i , define a point $p_i = (A_i[1], \dots, A_i[d]) \in \mathbb{R}^d$. For each B_j , define a set of $O(d^4)$ hyperplanes

$$H_j = \left\{ \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_u + B_j[v] = x_{u'} + B_j[v']\} \quad : \quad u, v, u', v' \in [d] \right\}.$$

Let P be the resulting set of m points and H be the resulting set of $O(d^4 m)$ hyperplanes. Apply the Cutting Lemma to H and P , and obtain sets H_Δ and P_Δ in time $d^{O(d)} m r^{O(d)}$. Take a cell Δ and an index $j \in [m]$. We describe how to handle the queries for K_{ij} for all $p_i \in P_\Delta$.

- CASE 1: some hyperplane of H_j is in H_Δ . Since $|H_\Delta|$ is small, there are not too many such j 's and we can afford to use a slow algorithm here:

Namely, for each $p_i \in P_\Delta$, just sort $A_i + B_j$. Then we can look up the answer for each $k \in K_{ij}$ in constant time. The total time is $O(\sum_{p_i \in P_\Delta} |K_{ij}| + |P_\Delta| \cdot d^2 \log d)$.

Since there are at most $|H_\Delta| = O(d^4 m/r)$ choices of j for each Δ , the total time over all Δ and all j is $O(Q + (d^4 m/r) \cdot m \cdot d^2 \log d)$.

- CASE 2: no hyperplane of H_j is in H_Δ . We can save time here by observing that the sorted ordering of $A_i + B_j$ is the same as the sorted ordering of $A_{i'} + B_j$ for every $p_i, p_{i'} \in P_\Delta$, because no hyperplane of H_j crosses Δ . In particular, the index pair for the k -th smallest in $A_i + B_j$ is the same as the index pair for the k -th smallest in $A_{i'} + B_j$ for every $p_i, p_{i'} \in P_\Delta$.

So, pick one representative point p_{i_Δ} in P_Δ , and just sort $A_{i_\Delta} + B_j$. Then we can look up the answer for each $k \in K_{ij}$ in constant time. The total time is $O(\sum_{p_i \in P_\Delta} |K_{ij}| + |P_\Delta| + d^2 \log d)$.

Since there are $d^{O(d)} r^{O(d)}$ cells Δ , the total time over all Δ and all $j \in [m]$ is $O(Q + m^2 + (dr)^{O(d)} \cdot m \cdot d^2 \log d)$.

The overall running time is $O(Q + m^2 + d^6(m^2/r) \log d + (dr)^{O(d)} m)$. Setting $r := d^7$ gives the desired result for $d \leq \delta \log m / \log \log m$. \square

It follows that $T_{\text{search},+}(m, Q, d) = O((Q + m^2) \log d)$ for $d = \delta \log m / \log \log m$, and so by Lemma 3.3, we immediately obtain a 3SUM algorithm with running time $O((n^2/d) \log d) = O((n^2/\log n)(\log \log n)^2)$. (Note that our solution here is simpler than previous solutions of similar running time [19, 21].)

To improve Theorem 3.5 by about a w factor, we use bit-packing techniques. (Note that such an improvement does not seem to work for the previous solutions [19, 21] that were based

on dominance instead of cuttings.) In Problem 3.2, note that each query list K_{ij} can be stored compactly in $O((|K_{ij}| \log d)/w + 1)$ words. So can the output for K_{ij} : if the k -th smallest of $A_i + B_j$ is $A_i[u] + B_j[v]$, it can be represented as an index pair $(u, v) \in [d]^2$. Thus, the total input/output size is $O((Q \log d)/w + m^2)$ in words.

We will actually use a variant of the input/output representation which lowers the m^2 term: Divide $[m]$ into m/\bar{w} blocks of \bar{w} consecutive indices for a fixed value $\bar{w} \geq w$. For each $i \in [m]$ and block β , we store a list $K_{i\beta}$ containing $(j \bmod \bar{w}, k)$ over all $j \in \beta$ and $k \in K_{ij}$. The input/output size for this \bar{w} -block representation becomes $O((Q \log(d\bar{w}))/w + m^2/\bar{w})$ in words.

Theorem 3.6. (Refined Version) $T_{\text{select},+}(m, d, Q) = O((Q \log^2(d\bar{w}))/w + m^2/\bar{w})$ for $d \leq \delta \log m / \log(\bar{w} \log m)$ for a sufficiently small constant $\delta > 0$.

Proof. We modify the proof of Theorem 3.5. Define H, P , and the cutting as before.

Take a cell Δ and a block β of \bar{w} consecutive indices in $[m]$. We describe how to handle the queries for $K_{i\beta}$ for all $p_i \in P_\Delta$.

- CASE 1: some hyperplane of $\bigcup_{j \in \beta} H_j$ is in H_Δ .

Take a fixed $p_i \in P_\Delta$. Sort $A_i + B_j$ for all $j \in \beta$ in $O(\bar{w}d^2 \log d)$ time. Create a table $f : \{0, \dots, \bar{w} - 1\} \times [d]^2 \rightarrow [d]^2$ where $f[j \bmod \bar{w}, k]$ stores the index pair for the k -th smallest of $A_i + B_j$. Look up f for the answers for $K_{i\beta}$ in $O((|K_{i\beta}| \log^2(d\bar{w}))/w + \bar{w}d^2)$ time by Fact 2.1(b). The total time over all $p_i \in P_\Delta$ is $O(\sum_{i \in P_\Delta} |K_{i\beta}| \log^2(d\bar{w}))/w + |P_\Delta| \cdot \bar{w}d^2 \log d$.

Since there are at most $|H_\Delta| = O(d^4 m/r)$ choices of β for each Δ , the total time over all Δ and all β is $O((Q \log^2(d\bar{w}))/w + (d^4 m/r) \cdot m \cdot \bar{w}d^2 \log d)$.

- CASE 2: no hyperplane of $\bigcup_{j \in \beta} H_j$ is in H_Δ .

Pick one representative point p_{i_Δ} in P_Δ . Sort $A_{i_\Delta} + B_j$ for all $j \in \beta$ in $O(\bar{w}d^2 \log d)$ time, and create a table $f : \{0, \dots, \bar{w} - 1\} \times [d]^2 \rightarrow [d]^2$ where $f[j \bmod \bar{w}, k]$ stores the index pair for the k -th smallest of $A_{i_\Delta} + B_j$. Concatenate the lists $K_{i\beta}$ over all $p_i \in P_\Delta$, look up f for the answers for the combined list in $O(\sum_{p_i \in P_\Delta} (|K_{i\beta}| \log^2(d\bar{w}))/w + \bar{w}d^2 + |P_\Delta|)$ time by Fact 2.1(b), then split the output list to get the answers for each $K_{i\beta}$.

Since there are $d^{O(d)} r^{O(d)}$ cells Δ , the total time over all Δ and m/\bar{w} blocks β is $O((Q \log^2(d\bar{w}))/w + (dr)^{O(d)} \cdot (m/\bar{w}) \cdot \bar{w}d^2 \log d + (m/\bar{w}) \cdot m)$.

The overall running time is at most $O((Q \log^2(d\bar{w}))/w + m^2/\bar{w} + d^6 \bar{w} (m^2/r) \log d + (dr)^{O(d)} m)$. Setting $r := d^7 \bar{w}^2$ gives the desired result for $d \leq \delta \log m / \log(\bar{w} \log m)$. \square

Unfortunately, the above theorem in itself is not sufficient to yield a further improvement to 3SUM, because of the $O(n^2/d)$ term in Lemma 3.3, unless we could choose a bigger d . We will propose more sophisticated bit-packing tricks to get around this bottleneck.

3.3 Batched $A + B$ Comparisons/Sorting via Bit Packing and Fredman's Trick

First we need a subroutine for solving the following subproblem:

Problem 3.7. (Batched $A + B$ Comparisons) *Given lists ("groups") $A_1, \dots, A_m, B_1, \dots, B_m$ of d real numbers each, and given "query" lists Q_{ij} of quadruples in $[d]^4$ with $\sum_{i,j} |Q_{ij}| = Q$, test whether $A_i[u] + B_j[v] \leq A_i[u'] + B_j[v']$ for each $(u, v, u', v') \in Q_{ij}$ and each $i, j \in [m]$.*

Let $T_{\text{comp},+}(m, d, Q)$ be the time complexity of this problem. Note that by bit packing, the total input size is $O((Q \log d)/w + m^2)$ in words, and the total output size is $O(Q/w + m^2)$ in words, since the output to each Q_{ij} is a $|Q_{ij}|$ -bit vector. (We will not need block representations here.) The following theorem is inspired by [11, Theorem 2.4] (which in turn was based on some ideas from an APSP algorithm of Han [23]):

Theorem 3.8. $T_{\text{comp},+}(m, d, Q) = O((Q \log^2(dw))/w + m^2)$ for $m \geq d^2 \log d$.

Proof. We use “Fredman’s trick”, i.e., the (trivial) observation that

$$A_i[u] + B_j[v] \leq A_i[u'] + B_j[v'] \iff A_i[u] - A_i[u'] \leq B_j[v'] - B_j[v].$$

Sort the $O(d^2m)$ elements

$$\{A_i[u] - A_i[u'] : i \in [m], u, u' \in [d]\} \cup \{B_j[v'] - B_j[v] : j \in [m], v, v' \in [d]\}$$

in $O(d^2m \log(dm))$ time. Let L be the resulting list. Create tables $f, g : [m] \times [d]^2 \rightarrow [d^2m]$ where $f[i, u, u']$ stores the rank of $A_i[u] - A_i[u']$ in L and $g[j, v', v]$ stores the rank of $B_j[v'] - B_j[v]$ in L .

Map each quadruple (u, u', v, v') in the list Q_{ij} to (i, j, u, u', v, v') . Concatenate the lists over all $(i, j) \in [m]^2$ in lexicographical order. In the concatenated list, map (i, j, u, u', v, v') to $(f[i, u, u'], j, v, v')$, and then to $(f[i, u, u'], g[j, v, v'])$, by two successive applications of Fact 2.1(b) in $O((Q \log^2(dm w))/w + d^2m)$ time. We can then obtain the output list of Boolean values corresponding to whether $f[i, u, u'] \leq g[j, v', v]$, and split to get the answers to each Q_{ij} . The total time is $O((Q \log^2(dm w))/w + m^2 + d^2m \log(dm)) = O((Q \log^2(dm w))/w + m^2)$ for $m \geq d^2 \log d$.

The $\log(dm w)$ factors in the above time bound can be replaced by $\log(dw)$ easily: Let $m_0 := d^2 \log d$. Divide $A_1, \dots, A_m, B_1, \dots, B_m$ into $O(m/m_0)$ blocks of $O(m_0)$ lists each. Apply the above algorithm to each pair of blocks. The total time is now $O((Q \log^2(dm_0 w))/w + (m/m_0)^2 \cdot m_0^2) = O((Q \log^2(dw))/w + m^2)$. \square

The above subroutine enables us to solve the following, tougher subproblem:

Problem 3.9. (Batched $A + B$ Sorting) *Given lists (“groups”) $A_1, \dots, A_m, B_1, \dots, B_m$ of d real numbers each, and given “query” lists Q_{ij} with $|Q_{ij}| \leq (d\ell)^{O(1)}$ and $\sum_{i,j} |Q_{ij}| = Q$, where each element of Q_{ij} is a sequence of ℓ pairs in $[d]^2$, sort each sequence $\langle (u_1, v_1), \dots, (u_\ell, v_\ell) \rangle$ in Q_{ij} so that at the end, $A_i[u_1] + B_j[v_1] \leq \dots \leq A_i[u_\ell] + B_j[v_\ell]$, for each $i, j \in [m]$.*

Let $T_{\text{sort},+}(m, d, \ell, Q)$ be the time complexity of this problem. Note that by bit packing, the total input/output size is $O((\ell Q \log d)/w + m^2)$ in words.

Theorem 3.10. $T_{\text{sort},+}(m, d, \ell, Q) = O(((\ell Q \log^2(d\ell w))/w + Q + m^2) \cdot \log \ell)$ for $m \geq \ell^{3/2} d^2 \log(d\ell)$.

Proof. For each $r \in [\ell]$, first define the list $L_{ij}^{(r)}$, where $L_{ij}^{(r)}[t]$ is the r -th pair of $Q_{ij}[t]$ for all $t \in [|Q_{ij}|]$. To generate these lists, map the r -th pair of $Q_{ij}[t] = (u, v)$ to the quadruple (u, v, r, t) , sort all quadruples in Q_{ij} by (r, t) (by Fact 2.1(a)), and split into sublists by r . The total time is $O((\ell Q \log^2(d\ell))/w + \ell m^2)$.

The main idea is to simulate a *sorting network* on ℓ elements, to sort $L_{ij}^{(1)}[t], \dots, L_{ij}^{(\ell)}[t]$ simultaneously for all t . Consider one round of the network. If it performs a compare-and-exchange

operation between the r -th and r' -th element, scan the lists $L_{ij}^{(r)}$ and $L_{ij}^{(r')}$, form a list of quadruples (u, v, u', v') for each $L_{ij}^{(r)}[t] = (u, v)$ and $L_{ij}^{(r')}[t] = (u', v')$; concatenate the resulting lists over all $O(\ell)$ compare-and-exchange operations in the round and get a single list per (i, j) ; apply the algorithm for batched $A + B$ comparisons with total query list size $O(\ell Q)$; then split the output to get the answers to the comparisons for each pair (r, r') ; finally execute the compare-and-exchange operations for each pair (r, r') by scanning the lists $L_{ij}^{(r)}$ and $L_{ij}^{(r')}$. Each round requires total time $O(T_{\text{comp},+}(m, d, \ell Q) + (\ell Q \log^2(d\ell))/w + \ell m^2) = O((\ell Q \log^2(d\ell w))/w + \ell m^2)$. By using the AKS sorting network [3], the number of rounds is $O(\log \ell)$.

At the end, concatenate $L_{ij}^{(r)}$ over all r into a single list L_{ij} , and sort by t , and split to get back the sequences $Q_{ij}[t]$. The total running time is $O(((\ell Q \log^2(d\ell w))/w + Q + \ell m^2) \cdot \log \ell)$ for $m \geq d^2 \log d$.

The ℓm^2 term in the above time bound can be easily reduced: Regroup $A_1, \dots, A_m, B_1, \dots, B_m$ into $O(m/\sqrt{\ell})$ lists of size $\sqrt{\ell}d$, by concatenating $\sqrt{\ell}$ lists into one. Then apply the above algorithm with m and d replaced by $m/\sqrt{\ell}$ and $\sqrt{\ell}d$. The query input/output lists can be reformatted in $O((\ell Q \log(d\ell))/w + m^2)$ time. The total time is $O(((\ell Q \log^2(d\ell w))/w + Q + \ell(m/\sqrt{\ell})^2) \cdot \log \ell + m^2)$ for $m/\sqrt{\ell} \geq (\sqrt{\ell}d)^2 \log(d\ell)$. \square

Note that the above algorithm can sort Q_{ij} even when extra fields are attached to each pair in Q_{ij} , provided that the extra fields require $O(\log(d\ell))$ bits.

3.4 Putting Everything Together via 2-Level Grouping

We now solve the batched $A + B$ searching problem for a bigger group size d near $w \log m$, by reducing to batched $A + B$ selection for a smaller group size:

Lemma 3.11. $T_{\text{search},+}(m, d, Q) = O((T_{\text{select},+}(\ell m, d/\ell, \ell Q) + T_{\text{sort},+}(m, d, \ell, Q) + \ell Q/w + Q + m^2 + \ell^2 m^2/\bar{w}) \cdot \log^{O(1)}(d\bar{w}) + dm \log d)$ for any given $\ell \leq \min\{d, \bar{w}\}$.

Proof. We may assume that ℓ and d are powers of 2. If $|C_{ij}| \geq d^2$, we can afford to use a slow algorithm: compute $A_i + B_j$ in $O(d^2 \log d)$ time and test for each $c \in C_{ij}$ whether $c \in A_i + B_j$ by binary search. The number of such C_{ij} 's is at most $O(Q/d^2)$ and so the total time for this step is $O((Q/d^2) \cdot d^2 \log d + Q \log d) = O(Q \log d)$. Thus, we may assume that $|C_{ij}| < d^2$ from now on.

Sort each A_i and B_j . Divide each A_i into sublists $A_{i1}, \dots, A_{i\ell}$ of size $d_0 := d/\ell$, and each B_j into sublists $B_{j1}, \dots, B_{j\ell}$ of size d_0 .

Consider a fixed $c \in C_{ij}$. We describe an algorithm $\text{search}(c, A_i, B_j)$ to find the predecessor of c in $A_i + B_j$. Observe that if the answer is in $A_{ip} + B_{jq}$, then we either have (i) $c \in [A_{ip}[1] + B_{jq}[1], A_{ip}[1] + B_{j,q+1}[1])$, or (ii) $c \in [A_{ip}[1] + B_{j,q+1}[1], A_{i,p+1}[1] + B_{j,q+1}[1])$. We assume the first case; the second case can be handled by a symmetric algorithm. The algorithm is given by the pseudocode below, and works as follows: For each $p \in [\ell]$, first find the unique index q_p for which $c \in [A_{ip}[1] + B_{jq_p}[1], A_{ip}[1] + B_{j,q_p+1}[1])$, by binary search (lines 2–3). Then find the predecessor of c in $A_{ip} + B_{jq_p}$ by another binary search, this time, over the ranks (lines 4–6), using an oracle for selection in $A_{ip} + B_{jq_p}$. The largest of the predecessors found gives us the overall predecessor of c in $A_i + B_j$.

$\text{search}(c, A_i, B_j)$:

1. initialize $q_1 = \dots = q_\ell = k_1 = \dots = k_\ell = 1$

2. for $s = \ell/2, \ell/4, \dots, 1$:
3. for each $p \in [\ell]$, if $c > A_{ip}[1] + B_{j, q_p+s}[1]$ then $q_p := q_p + s$
4. for $s = d_0^2/2, d_0^2/4, \dots, 1$:
5. for each $p \in [\ell]$, if $c > (\text{the } (k_p + s)\text{-th smallest in } A_{ip} + B_{j, q_p})$ then $k_p := k_p + s$
6. return $\max\{\text{the } k_p\text{-th smallest in } A_{ip} + B_{j, q_p} : p \in [\ell]\}$

The main idea is to run algorithm $\text{search}(c, A_i, B_j)$ simultaneously for all $c \in C_{ij}$ and all $i, j \in [m]$, using bit-packed lists. For each i, j, c , we maintain a list L_{ijc} of $O(\ell)$ tuples (p, q_p, k_p) for all $p \in [\ell]$ and variables q_p and k_p kept by the algorithm. These lists L_{ijc} require a total of $O((\ell Q \log d)/w + Q)$ words.

To implement line 3, we first sort the tuples (p, q_p, k_p) in L_{ijc} by $A_{ip}[1] + B_{j, q_p}[1]$; by the batched $A + B$ sorting subroutine, this takes $O(T_{\text{sort},+}(m, d, \ell, Q))$ total time over all i, j, c . After sorting, we can resolve all comparisons between c and $A_{ip}[1] + B_{j, q_p}[1]$ for all $p \in [\ell]$, by standard binary search for c in $O(\log \ell)$ time per i, j, c (no bit packing needed here). The total time for all these binary searches is $O(Q \log d)$.

Line 5 (or 6) is similar. However, before we can sort, we need to find the index pair for the $(k_p + s)$ -th smallest in $A_{ip} + B_{j, q_p}$ for all $p \in [\ell]$. This can be done by the batched $A + B$ selection algorithm in $O(T_{\text{select},+}(\ell m, d_0, \ell Q))$ total time over all i, j, c . Before the call to batched $A + B$ selection, some setup is required to reformat to a \bar{w} -block input representation: first map each tuple (p, q_p, k_p) in L_{ijc} to $(p, q_p, k_p, j \bmod (\bar{w}/\ell), c)$ (note that c can be encoded as an integer in $[|C_{ij}|] = [O(d^2)]$); for each i and each block β of \bar{w}/ℓ consecutive indices in $[m]$, let $L_{i\beta}$ be the concatenation of L_{ijc} over all c and all $j \in \beta$; sort $L_{i\beta}$ by p (by Fact 2.1(a)), and split it into sublists $L_{ip\beta}$ with a common p ; then we are ready for the call. After the call, we can concatenate the answers for $L_{ip\beta}$ over all p to get the answers for $L_{i\beta}$, then sort $L_{i\beta}$ by $(j \bmod (\bar{w}/\ell), c)$ to get back the answers for L_{ijc} . Since there are $O(m\ell/\bar{w})$ choices for β and so $O(m \cdot \ell \cdot m\ell/\bar{w})$ choices for (i, p, β) , these extra steps take $O((\ell Q \log^2 \bar{w})/w + \ell^2 m^2/\bar{w})$ total time.

Since the entire algorithm has $\log \ell + \log(d_0^2)$ iterations, the overall running time is multiplied by an $O(\log d)$ factor. \square

Combining Lemma 3.11 and Theorems 3.6 and 3.10, we obtain

$$\begin{aligned} T_{\text{search},+}(m, d, Q) &= O((\ell Q/w + Q + m^2 + \ell^2 m^2/\bar{w}) \log^{O(1)}(d\bar{w})) \\ &= O((Q + m^2) \log^{O(1)} w) \end{aligned}$$

by setting $\ell \approx (1/\delta)w$ and $\bar{w} := w^2$, assuming that $d \leq w \log m / \log w$, and $w = \Omega(\log m)$, and $w \leq m^{o(1)}$.

By Lemma 3.3 with $d \approx w \log n / \log w$, we obtain our main result:

Corollary 3.12. *3SUM can be solved in $O((n^2/(w \log n)) \log^{O(1)} w) \leq O((n^2/\log^2 n)(\log \log n)^{O(1)})$ time, assuming that $w = \Omega(\log n)$ and $w \leq n^{o(1)}$.*

We do not feel it is important to optimize the $\log \log$ factors, but for those interested, the above 3SUM algorithm has 5 $\log \log n$ factors. If we do not care about $\log \log n$ factors, we can replace the AKS sorting network by one of Batcher's sorting networks, which is simpler. With randomization, sorting networks can probably be avoided entirely (by using approximate medians instead of sorting).

4 (select,+)-Convolution/Matrix Multiplication

We can apply similar ideas to the (select,+)-matrix multiplication and (select,+)-convolution problems. For two lists A and B of d elements, let $A \hat{+} B$ denote the list $\langle A[1] + B[1], A[2] + B[2], \dots, A[d] + B[d] \rangle$. Let $T_{\text{select}, \hat{+}}(m, d, Q)$ be the time complexity of batched $A \hat{+} B$ selection, i.e., the variant of Problem 3.2 with $+$ replaced by $\hat{+}$.

4.1 Reduction to Batched $A \hat{+} B$ Selection

We first show how (select,+)-matrix multiplication and (select,+)-convolution can be reduced to batched $A \hat{+} B$ selection for small group size d .

Lemma 4.1.

- (a) *(select,+)-matrix multiplication can be solved in $O((T_{\text{select}, \hat{+}}(n, d, n^2) + n^2) \cdot (n/d) \log d)$ time.*
- (b) *(select,+)-convolution can be solved in $O((T_{\text{select}, \hat{+}}(n/d, d, n^2/d^2) + n^2/d^2) \cdot d \log d)$ time.*

Proof. Given ℓ sorted arrays X_1, \dots, X_ℓ of d elements each, there are known algorithms that can find the k -th smallest in $O(\ell \log d)$ time [17, 18]. Such algorithms proceed in $O(\log d)$ iterations, where each iteration takes $O(\ell)$ time and requires looking up $O(1)$ entries per array in parallel. The exact details are not important for our purposes, but for the sake of completeness, we provide one such algorithm:

select(k, X_1, \dots, X_ℓ):

1. initialize $k_1 = \dots = k_\ell = 0$ and $s_1 = \dots = s_p = d$
2. while $S := s_1 + \dots + s_\ell > 3\ell$ do:
3. if $k \leq S/2$:
4. for each $p \in [\ell]$, let $\mu_p := X_p[k_p + \lfloor s_p/3 \rfloor]$ and $\nu_p := X_p[k_p + 2 \lfloor s_p/3 \rfloor]$
5. let x be the smallest such that $\sum_{\mu_p \leq x} \lfloor s_p/3 \rfloor + \sum_{\nu_p \leq x} \lfloor s_p/3 \rfloor > k$
6. for each $p \in [\ell]$, if $\mu_p \geq x$ then $s_p := \lfloor s_p/3 \rfloor$ else if $\nu_p \geq x$ then $s_p := 2 \lfloor s_p/3 \rfloor$
7. else negate and reverse $X_p[k_p + 1, \dots, k_p + s_p]$ for each $p \in [\ell]$, and set $k := S - k + 1$
8. return the k -th smallest in $\bigcup_{p \in [\ell]} X_p[k_p + 1, \dots, k_p + s_p]$

The above algorithm maintains the invariant that the answer is the k -th smallest in $\bigcup_{p \in [\ell]} X_p[k_p + 1, \dots, k_p + s_p]$. Consider the case when $k \leq S/2$. In line 5, the answer is at most x , because there are more than k elements at most x . Thus, in line 6, if $\mu_p \geq x$, we can remove the top two-thirds of $X_p[k_p + 1, \dots, k_p + s_p]$, else if $\nu_p \geq x$, we can remove the top third. The number of remaining elements is at most $\sum_{\mu_p < x} \lfloor s_p/3 \rfloor + \sum_{\nu_p < x} \lfloor s_p/3 \rfloor + \sum_{p=1}^{\ell} (s_p - 2 \lfloor s_p/3 \rfloor) \leq k + S/3 + 2\ell \leq 5S/6 + 2\ell$. The case when $k \geq S/2$ is symmetric (note that the negate-and-reverse operation can be done implicitly). It takes at most $O(\log d)$ iterations for S to drop from $d\ell$ to below 3ℓ . Lines 5 and 8 require $O(\ell)$ time by using a linear-time (weighted) selection algorithm, so the total time is indeed $O(\ell \log d)$.

Now we are ready to prove the lemma:

- (a) Let $\ell := n/d$. Divide the i -th row of A into sublists $A_i^{(1)}, \dots, A_i^{(\ell)}$ of d elements each, and divide the j -th column of B into sublists $B_j^{(1)}, \dots, B_j^{(\ell)}$ of d elements each. For each $i, j \in [n]$,

we want to compute c_{ij} = the k_{ij} -th smallest of the union of $A_i^{(1)} \hat{+} B_j^{(1)}, \dots, A_i^{(\ell)} \hat{+} B_j^{(\ell)}$. Let $X_{ij}^{(p)}$ denote the list $A_i^{(p)} \hat{+} B_j^{(p)}$ rearranged in sorted order. The idea is to run the above algorithm $\text{select}(k_{ij}, X_{ij}^{(1)}, \dots, X_{ij}^{(\ell)})$ simultaneously for all $i, j \in [n]$. We do not explicitly store the sorted list $X_{ij}^{(p)}$, but when desired, can retrieve the k -th element in the sorted list for any given k by selection in $A_i^{(p)} \hat{+} B_j^{(p)}$. Thus, in each iteration, line 4 can be done by ℓ instances of batched $A \hat{+} B$ selection (one instance per p); the total time over all i, j is $O((T_{\text{select}, \hat{+}}(n, d, n^2) + n^2) \cdot \ell)$. Line 5 takes $O(\ell)$ time per i, j , for a total of $O(n^2 \ell)$ time. Since the algorithm has $O(\log d)$ iterations, the overall running time is multiplied by an $O(\log d)$ factor.

- (b) Divide A into $\ell := n/d$ sublists A_1, \dots, A_ℓ of size d each, and B into sublists B_1, \dots, B_ℓ of size d each. For each $i \in [\ell]$, we want to compute c_{id+1} = the k_{id+1} -th smallest of the union of $A_1 \hat{+} B_i^r, A_2 \hat{+} B_{i-1}^r, \dots, A_i \hat{+} B_1^r$, where L^r denotes the reverse of L . Let X_{ip} denote the list $A_p \hat{+} B_{i-p+1}^r$ rearranged in sorted order. The idea is to run the algorithm $\text{select}(k_i, X_{i1}, \dots, X_{ii})$ simultaneously for all $i \in [\ell]$. In each iteration, line 4 can be done by batched $A \hat{+} B$ selection; the total time over all i is $O(T_{\text{select}, \hat{+}}(\ell, d, \ell^2) + \ell^2)$. Line 5 takes $O(\ell)$ time per i , for a total of $O(\ell^2)$ time. Since the algorithm has $O(\log d)$ iterations, the overall running time is multiplied by an $O(\log d)$ factor.

We have computed only the output entries c_{id+1} . To compute c_{id+r} for other $r \in [d]$, we can shift A by $r - 1$ positions and apply the above algorithm. The final running time is multiplied by an $O(d)$ factor. \square

Note that $T_{\text{select}, \hat{+}}(m, d, Q) = O(T_{\text{select}, +}(m, d, Q))$, since batched $A \hat{+} B$ selection reduces to batched $A + B$ selection, for example, by mapping $A_i[p]$ to $pM + A_i[p]$ and $B_j[q]$ to $-qM + B_j[q]$ for a sufficiently large M , and mapping k to $d(d - 1)/2 + k$.

4.2 Putting Everything Together via 2-Level Grouping

We can further adapt Lemma 3.11 to solve not just batched $A \hat{+} B$ searching but batched $A \hat{+} B$ selection:

Lemma 4.2. $T_{\text{select}, \hat{+}}(m, d, Q) = O((T_{\text{select}, \hat{+}}(\ell m, d/\ell, \ell Q) + T_{\text{sort}, +}(m, d, \ell, Q) + \ell Q/w + Q + m^2 + \ell^2 m^2/\bar{w}) \cdot \log^{O(1)}(d\bar{w}) + dm \log d)$ for any given $\ell \leq \min\{d, \bar{w}\}$.

Proof. Divide each A_i into sublists $A_{i1}, \dots, A_{i\ell}$ of size $d_0 := d/\ell$, and each B_j into sublists $B_{j1}, \dots, B_{j\ell}$ of size d_0 . Let X_{ijp} denote the list $A_{ip} + B_{jp}$ rearranged in sorted order.

The main idea is to run the algorithm $\text{select}(k, X_{ij1}, \dots, X_{ij\ell})$, from the proof of Lemma 4.1, simultaneously for all $k \in K_{ij}$ and $i, j \in [m]$, using bit-packed lists. For each i, j, k , we maintain a list L_{ijk} of $O(\ell)$ tuples (p, k_p, s_p) of tuples for all $p \in [\ell]$ and the variables k_p and s_p kept by the algorithm. We do not explicitly store the sorted list X_{ijp} , but when desired, can retrieve the k -th element in the sorted list for any given k by selection in $A_{ip} \hat{+} B_{jp}$. For line 4, we apply batched $A \hat{+} B$ selection, taking $O(T_{\text{select}, \hat{+}}(\ell m, d_0, \ell Q))$ total time. For line 5, we apply batched $A + B$ sorting to first sort the tuples in L_{ijk} by μ_p and by ν_p , which would then make x easy to find; this takes $O(T_{\text{sort}, +}(m, d, \ell, Q))$ total time. For line 8, we again use batched $A + B$ sorting. Other details of the bit-packed list manipulation and running-time analysis are as in the proof of Lemma 3.11. \square

As before, we obtain $T_{\text{select}, \dagger}(m, d, Q) = O((Q + m^2) \log^{O(1)} w)$ assuming that $d \leq w \log m / \log w$, and $w = \Omega(\log m)$, and $w \leq m^{o(1)}$.

By Lemma 4.1 with $d \approx w \log n / \log w$, we obtain:

Corollary 4.3. *(median, +)-matrix multiplication can be solved in $O((n^3/(w \log n)) \log^{O(1)} w) \leq O((n^3/\log^2 n)(\log \log n)^{O(1)})$ time, and (median, +)-convolution can be solved in $O((n^2/(w \log n)) \log^{O(1)} w) \leq O((n^2/\log^2 n)(\log \log n)^{O(1)})$ time, assuming that $w = \Omega(\log n)$ and $w \leq n^{o(1)}$.*

5 Algebraic 3SUM

We can generalize our 3SUM algorithms to solve Barba et al.'s algebraic 3SUM problem [6] stated in the introduction. Define $A +_{\varphi} B = \{\varphi(a, b) : a \in A, b \in B\}$. Define the batched $A +_{\varphi} B$ searching problem as in Problem 3.1, with $+$ replaced by \oplus , except that instead of finding the predecessor, we just want to whether $c \in A_i \oplus B_j$ for each $c \in C_{ij}$. Define the batched $A +_{\varphi} B$ selection/comparison/sorting problems as in Problems 3.2–3.9, with $+$ replaced by $+_{\varphi}$. Let $T_{\text{search}, +_{\varphi}}(\cdot), T_{\text{select}, +_{\varphi}}(\cdot), T_{\text{comp}, +_{\varphi}}(\cdot), T_{\text{sort}, +_{\varphi}}(\cdot)$ be the time complexities of these problems.

5.1 Reduction to Batched $A +_{\varphi} B$ Searching/Selection

Lemma 5.1. *Algebraic 3SUM can be solved in time $O(T_{\text{search}, +_{\varphi}}(n/d, d, n^2/d) + n^2/d + n \log n)$ for any given $d \leq n$.*

Proof. As in the proof of Lemma 3.3. We need the property that for any fixed c , the number of (i, j) pairs with $c \in \varphi([A_i[1], A_{i+1}[1]] \times [B_j[1], B_{j+1}[1]])$ is $O(n/d)$ (and that these (i, j) pairs can be found in $O(n/d)$ time). In other words, a curve of the form $\{(x, y) \in \mathbb{R}^2 : \varphi(x, y) = c\}$ can intersect at most $O(n/d)$ grid cells formed by the grid lines $x = A_i[1]$ and $y = B_j[1]$. This was noted by Barba et al. [6] and follows since the curve can intersect each of the $O(n/d)$ grid lines at most $O(1)$ times. One extra case needs to be addressed: the curve may be completely contained in a grid cell without intersecting the boundary grid lines. But the number of such grid cells is $O(1)$ since the curve has $O(1)$ connected components. \square

5.2 Batched $A +_{\varphi} B$ Selection via Cuttings in Near-Logarithmic Dimensions

To adapt the proofs of Theorems 3.5 and 3.6, we replace the hyperplanes of H_j with (hyper)surfaces $\{(x_1, \dots, x_d) \in \mathbb{R}^d : \varphi(x_u, B_j[v]) = \varphi(x_{u'}, B_j[v'])\}$ for each $u, v, u', v' \in [d]$. For these surfaces, we can apply the following version of the Cutting Lemma (Fact 3.4) with $t = 2$:

Fact 5.2. (Generalized Cutting Lemma) *Fact 3.4 remains true for a set H of n surfaces in \mathbb{R}^d each of which is of the form $\{(x_1, \dots, x_d) \in \mathbb{R}^d : (x_{u_1}, \dots, x_{u_t}) \in S\}$ for some semi-algebraic set S of constant degree, for some $u_1, \dots, u_t \in [d]$, and for some absolute constant t .*

Proof. Let H_{u_1, \dots, u_t} be the subset of all surfaces in H with a common tuple (u_1, \dots, u_t) . For each fixed $(u_1, \dots, u_t) \in [d]^t$, we can compute a cutting Γ_{u_1, \dots, u_t} into $r^{O(t)}$ cells, so that the relative interior of each cell intersects at most $|H_{u_1, \dots, u_t}|/r$ surfaces of H_{u_1, \dots, u_t} , by known results from computational geometry in a constant dimension t [2]. We output the overlay Γ of all these $O(d^t)$ cuttings, i.e., a cell in Γ is the intersection of $O(d^t)$ cells from the Γ_{u_1, \dots, u_t} 's. Clearly, each cell in Γ intersects at most n/r surfaces of H .

Note that the cells of Γ are the cells in an arrangement of $O(d^t r^{O(t)})$ semi-algebraic sets in \mathbb{R}^d . Thus, the complexity of Γ is $O((d^t r^{O(t)})^d) \leq d^{O(d)} r^{O(d)}$, assuming that t is a constant.

In our applications, we do not need an explicit representation of the cells of Γ (and they do not necessarily have constant complexity). Given the point set P , we can assign each point $p \in P$ to the cell in Γ_{u_1, \dots, u_t} containing p in $O(\log r)$ time by t -dimensional point location for each tuple (u_1, \dots, u_t) . From this, we obtain the label of the cell in Γ containing p . \square

Theorem 5.3. $T_{\text{select}, +\varphi}(m, d, Q) = O((Q \log^2(d\bar{w}))/w + m^2/\bar{w})$ for $d \leq \delta \log m / \log(\bar{w} \log m)$ for a sufficiently small constant $\delta > 0$.

Proof. As in the proof of Theorem 3.6, using Fact 5.2. \square

The analog of Theorem 3.5 without bit packing already yields an algorithm for algebraic 3SUM in $O((n^2/\log n)(\log \log n)^2)$ time, which is faster, and somewhat simpler, than Barba et al.'s algorithm. We next adapt the ideas in Section 3.3.

5.3 Batched $A +_{\varphi} B$ Comparisons/Sorting via Bit Packing and Fredman's Trick

Theorem 5.4. $T_{\text{comp}, +\varphi}(m, d, Q) = O((Q \log^2(dw))/w + m^2)$ for $m \geq d^{10} \log^2 d$.

Proof. We adapt the proof of Theorem 3.8 as follows. Consider a block β of r consecutive indices in $[m]$. Build the two-dimensional arrangement \mathcal{A} of $O(rd^2)$ curves $\{(x, x') \in \mathbb{R}^2 : \varphi(x, B_j[v]) = \varphi(x', B_j[v'])\}$ over all $j \in \beta$ and $v, v' \in [d]$. The arrangement has $O((rd^2)^2) = O(r^2 d^4)$ cells, and we can build a data structure in $O(r^2 d^4 \log^{O(1)} d)$ time supporting point location queries in \mathcal{A} in $O(\log d)$ time [2]. Create a table $f : [m] \times [d]^2 \rightarrow [O(r^2 d^4)]$ that maps (i, u, u') to the index of the cell of \mathcal{A} that contains the point $(A_i[u], A_i[u'])$. The table can be constructed in $O(md^2 \cdot \log d)$ time by repeated point location queries. Next create a table $g : [O(r^2 d^4)] \times \beta \times [d]^2 \rightarrow \{\text{true}, \text{false}\}$ that maps (γ, j, v, v') to whether $\varphi(x, B_j[v]) \leq \varphi(y, B_j[v'])$ for an arbitrary point (x, y) in the cell γ of \mathcal{A} . The table can be constructed in $O(r^2 d^4 \cdot rd^2) = O(r^3 d^6)$ time. Map each tuple (u, v, u', v') in Q_{ij} to (i, j, u, v, u', v') . Concatenate the Q_{ij} 's over all $(i, j) \in [m] \times \beta$. In the concatenated list, map (i, j, u, v, u', v') to $(f[i, u, u'], j, v, v')$, and then to $g[f[i, u, u'], j, v, v']$, by two successive applications of Fact 2.1(b) in $O(\sum_i \sum_{j \in \beta} (|Q_{ij}| \log^2(dmw))/w + md^2 + r^3 d^6)$ time. We can then split the resulting list to get the answers to each Q_{ij} for all $(i, j) \in [m] \times \beta$.

Repeating the process for all $O(m/r)$ blocks β gives total time $O((Q \log^2(dmw))/w + (m/r) \cdot (md^2 \log d + r^3 d^6) + m^2) = O((Q \log^2(dmw))/w + m^2)$ by setting $r = d^2 \log d$, assuming that $m \geq d^{10} \log^2 d$.

As before, the $\log(dmw)$ factors can be replaced by $\log(dw)$. \square

Theorem 5.5. $T_{\text{sort}, +\varphi}(m, d, \ell, Q) = O(((\ell Q \log^2(d\ell w))/w + Q + m^2) \cdot \log \ell)$ for $m \geq \ell^{11/2} d^{10} \log^2(d\ell)$.

Proof. As in the proof of Theorem 3.10. \square

5.4 Putting Everything Together via 2-Level Grouping

Lemma 5.6. $T_{\text{search}, +\varphi}(m, d, Q) = O((T_{\text{select}, +\varphi}(\ell m, d/\ell, \ell Q) + T_{\text{sort}, +\varphi}(m, d, \ell, Q) + \ell Q/w + Q + m^2 + \ell^2 m^2/\bar{w}) \cdot \log^{O(1)}(d\bar{w}) + dm \log d)$ for any given $\ell \leq \min\{d, \bar{w}\}$.

Proof. As in the proof of Lemma 3.11. In algorithm $\text{search}(c, A_i, B_j)$, replace $+$ with φ in line 3. If $c \in A_{ip} +_{\varphi} B_{jq}$, then the curve $\{(x, y) \in \mathbb{R}^2 : \varphi(x, y) = c\}$ may hit the left boundary edge of $[A_{ip}[1], B_{jq}[1]] \times [A_{i,p+1}[1], B_{j,q+1}[1]]$, in which case the algorithm would be correct; it may also hit one of the other three boundary edges, but each of these cases can be handled by a similar algorithm. One extra case needs to be addressed: the curve may be completely contained in a grid cell. Here, we can pick an arbitrary point in each connected component of the curve and find the unique (p, q) pair for which $[A_{ip}[1], B_{jq}[1]] \times [A_{i,p+1}[1], B_{j,q+1}[1]]$ contains the point; there are only $O(1)$ extra (p, q) pairs to check for each $c \in C_{ij}$. \square

Corollary 5.7. *The algebraic 3SUM problem can be solved in $O((n^2/(w \log n)) \log^{O(1)} w) \leq O((n^2/\log^2 n)(\log \log n)^{O(1)})$ time, assuming that $w = \Omega(\log n)$ and $w \leq n^{o(1)}$.*

6 Offline Range Searching for Bichromatic Segment Intersections

We next adapt our 3SUM algorithms to solve the geometric problems (i)–(iv) stated in the introduction. It suffices to consider (iv) offline triangle range searching for bichromatic segment intersections, since the other three problems reduce to it.

We first concentrate on the special case where all line segments and triangles are *long*, i.e., have all endpoints lying on the boundary of a fixed triangle Δ_0 . We may assume that the long triangles in C are halfplanes, since we can replace each long triangle with its at bounding halfplanes, so that the number of intersection points outside the triangle is equal to the sum of the number of intersection points outside the halfplanes. Without loss of generality, we may assume that the halfplanes are upper halfplanes, and that all the segments in A and B touch a common edge e_0 of Δ_0 (the original problem reduces to 3 instances with this property). We may assume that e_0 is vertical.

For a point $a \in \mathbb{R}^2$, let a_x and a_y be its x - and y -coordinates respectively. For two points $a, b \in \mathbb{R}^2$, let $\lambda(a, b)$ denote the line through a and b , i.e., $\{(x, y) \in \mathbb{R}^2 : (b_y - a_y)x + (a_x - b_x)y = b_y a_x - b_x a_y\}$. The slope of $\lambda(a, b)$ is

$$\mu(a, b) := \frac{b_y - a_y}{b_x - a_x},$$

and the x -coordinate of the intersection of $\lambda(a, b)$ and $\lambda(a', b')$ is given by the formula

$$\xi(a, b, a', b') := \frac{(a'_x - b'_x)(b_y a_x - b_x a_y) - (a_x - b_x)(b'_y a'_x - b'_x a'_y)}{(a'_x - b'_x)(b_y - a_y) - (a_x - b_x)(b'_y - a'_y)}.$$

For a segment a , let $a^* \in \mathbb{R}^2$ denote the point dual to the line extension of a . For a halfplane c , let $c^* \in \mathbb{R}^2$ denote the point dual to the line bounding c . For a point a , let a^* denote the line dual to a . For a set S of points, let $S^* = \{a^* : a \in S\}$. For a set S of lines, let $\mathcal{A}(S)$ denote the arrangement of S .

For two lists A and B of long segments in \mathbb{R}^2 , define $A \oplus B$ as the set of all intersection points between A and B .

6.1 Reduction to Batched $A \oplus B$ Searching/Selection

We replace Problems 3.1 and 3.2 with the following:

Problem 6.1. (Batched $A \oplus B$ Searching) *Given lists $A_1, \dots, A_m, B_1, \dots, B_m$ of d long segments each, and given “query” lists C_{ij} of lower halfplanes with $\sum_{i,j} |C_{ij}| = Q$, count the number of points in $A_i \oplus B_j$ inside c , i.e., the number of lines in $\mathcal{A}((A_i \oplus B_j)^*)$ below the point c^* , for each $c \in C_{ij}$ and each $i, j \in [m]$.*

Problem 6.2. (Batched $A \oplus B$ Selection)

- (i) *Given lists $A_1, \dots, A_m, B_1, \dots, B_m$ of d long segments each, and given “query” lists K_{ij} of numbers in $[d^4]$ with $\sum_{i,j} |K_{ij}| = Q$, find the k -th leftmost vertex in $\mathcal{A}((A_i \oplus B_j)^*)$, for each $k \in K_{ij}$ and each $i, j \in [m]$.*
- (ii) *Given sorted lists $A_1, \dots, A_m, B_1, \dots, B_m$ of d long segments each, and given “query” lists K_{ij} of pairs in $[d^4] \times [d^2]$ with $\sum_{i,j} |K_{ij}| = Q$, find the k' -th lowest line in $\mathcal{A}((A_i \oplus B_j)^*)$ at the x -value of the k -th leftmost vertex, for each $(k, k') \in K_{ij}$ and each $i, j \in [m]$.*

Let $T_{\text{search}, \oplus}(m, d, Q)$ and $T_{\text{select}, \oplus}(m, d, Q)$ be the time complexities of Problems 6.1 and 6.2 respectively. Note that $T_{\text{search}, \oplus}(m, d, Q) = O((T_{\text{select}, \oplus}(m, d, Q) + Q + m^2) \cdot \log d)$ by simultaneous binary searches, first in x via Problem 6.2(i), then in y via Problem 6.2(ii), to find the level of c^* in $\mathcal{A}((A_i \oplus B_j)^*)$, for all $c \in C_{ij}$ and all i, j . (This is analogous to the standard *slab method* for planar point location [27].)

Lemma 6.3. *Offline halfplane range searching for bichromatic segment intersections in the long case can be solved in $O(T_{\text{search}, \oplus}(n/d, d, n^2/d) + n^2/d + n \log n)$ time for any given $d \leq n$.*

Proof. We adapt the proof of Lemma 3.3. Sort A and B by the y -values of their endpoints on e_0 . As before, divide A into sublists $A_1, \dots, A_{n/d}$ of size d , and B into sublists $B_1, \dots, B_{n/d}$ of size d . Recall that the segments in A (resp. B) are disjoint. Let α_i denote the region between $A_i[1]$ and $A_{i+1}[1]$ within Δ_0 , and let β_j denote the region between $B_j[1]$ and $B_{j+1}[1]$ within Δ_0 . For each $c \in C$, there are two types of intersections to count:

1. intersections in $A_i \oplus B_j$ where $\alpha_i \cap \beta_j$ intersects $\partial(c \cap \Delta_0)$, and
2. intersections in $A_i \oplus B_j$ where $\alpha_i \cap \beta_j$ is in the interior of $c \cap \Delta_0$.

For type-1 intersections, put c in C_{ij} iff α_i and β_j intersect along $\partial(c \cap \Delta_0)$. The number of such (i, j) pairs is $O(n/d)$ and they can be found in $O(n/d)$ time by a linear scan over the intersections of the segments $A_i[1]$'s and $B_j[1]$'s with $\partial(c \cap \Delta_0)$. This gives an instance of Problem 6.1 with total query list size $Q = O(n^2/d)$.

For type-2 intersections, it suffices to count the number of (i, j) pairs with $\alpha_i \cap \beta_j$ in the interior of $c \cap \Delta_0$, and multiply the number by d^2 ; the count can be computed in $O(n/d)$ time by another linear scan over the intersections of the segments $A_i[1]$'s and $B_j[1]$'s with $\partial(c \cap \Delta_0)$. \square

6.2 Batched $A \oplus B$ Selection via Cuttings in Near-Logarithmic Dimensions

Theorem 6.4. $T_{\text{select}, \oplus}(m, d, Q) = O((Q \log^2(d\bar{w}))/w + m^2/\bar{w})$ for $d \leq \delta \log m / \log(\bar{w} \log m)$ for a sufficiently small constant $\delta > 0$.

Proof. To adapt the proof of Theorems 3.5 and 3.6, we redefine $p_i = (A_i[1]_x^*, A_i[1]_y^*, \dots, A_i[d]_x^*, A_i[d]_y^*) \in \mathbb{R}^{2d}$, and redefine H_j as a set of $O(d^8)$ surfaces, containing

$$\{(x_1, y_1, \dots, x_d, y_d) \in \mathbb{R}^{2d} : \xi((x_u, y_u), B_j[v]^*, (x_{u'}, y_{u'}), B_j[v']^*) = \xi((x_{u''}, y_{u''}), B_j[v'']^*, (x_{u'''}, y_{u'''}) , B_j[v''']^*)\}$$

for every $u, v, u', v', u'', v'', u''', v''' \in [d]$, and also

$$\left\{ (x_1, y_1, \dots, x_d, y_d) \in \mathbb{R}^{2d} : \mu((x_u, y_u), B_j[v]^*) = \mu((x_{u'}, y_{u'}), B_j[v']^*) \right\}$$

for every $u, v, u', v' \in [d]$,

$$\left\{ (x_1, y_1, \dots, x_d, y_d) \in \mathbb{R}^{2d} : x_u = B_j[v]_x^* \right\}$$

for every $u, v \in [d]$, and

$$\left\{ (x_1, y_1, \dots, x_d, y_d) \in \mathbb{R}^{2d} : \mu((x_u, y_u), B_j[v]^*) = \mu(e^*, B_j[v]^*) \right\}$$

for every $u, v \in [d]$ and each of the three edges e of Δ_0 . Each of these surfaces can be re-expressed as $\{(x_1, y_1, \dots, x_d, y_d) \in \mathbb{R}^{2d} : P(x_u, y_u, x_{u'}, y_{u'}, x_{u''}, y_{u''}, x_{u'''}, y_{u'''}) = 0\}$ for some degree-4 polynomial P and $u, u', u'', u''' \in [d]$, so the generalized Cutting Lemma (Fact 5.2) is applicable with $t = 8$.

Constructing the arrangement $\mathcal{A}((A_i \oplus B_j)^*)$ costs $O(d^4)$ time instead of $O(d^2 \log d)$, but this does not affect the final bound.

In Case 2, due to the definition of H_j , the sorted x -ordering of the vertices of $\mathcal{A}((A_i \oplus B_j)^*)$ is the same as the sorted x -ordering of the vertices of $\mathcal{A}((A_{i'} \oplus B_j)^*)$, and the sorted ordering of the slopes of $\mathcal{A}((A_i \oplus B_j)^*)$ is the same as the sorted ordering of the slopes of $\mathcal{A}((A_{i'} \oplus B_j)^*)$, for every $p_i, p_{i'} \in P_\Delta$. It follows that the k -th leftmost vertex, or the k' -th lowest line at the x -value of the k -th leftmost vertex, is the same. The rest of the proof is as before. \square

As before, the above theorem leads to an $O((n^2/\log n)(\log \log n)^{O(1)})$ -time algorithm, which is already new. We next adapt the ideas in Section 3.3.

6.3 Batched $A \oplus B$ Comparisons/Sorting via Bit Packing and Fredman's Trick

We redefine Problem 3.7 as follows:

Problem 6.5. (Batched $A \oplus B$ Comparisons)

- (i) Given lists $A_1, \dots, A_m, B_1, \dots, B_m$ of d long segments each, and given “query” lists Q_{ij} of tuples in $[d]^8$ with $\sum_{i,j} |Q_{ij}| = Q$, test whether $\xi(A_i[u]^*, B_j[v]^*, A_i[u']^*, B_j[v']^*) \leq \xi(A_i[u'']^*, B_j[v'']^*, A_i[u''']^*, B_j[v''']^*)$ for each $(u, v, u', v', u'', v'', u''', v''') \in Q_{ij}$ and each $i, j \in [m]$.
- (ii) Given lists $A_1, \dots, A_m, B_1, \dots, B_m$ of d long segments each, and given “query” lists Q_{ij} of tuples in $[d]^4$ with $\sum_{i,j} |Q_{ij}| = Q$, test whether $\mu(A_i[u]^*, B_j[v]^*) \leq \mu(A_i[u']^*, B_j[v']^*)$ for each $(u, v, u', v') \in Q_{ij}$ and each $i, j \in [m]$.

Let $T_{\text{comp}, \oplus}(m, d, Q)$ be the time complexity of the above problem.

Theorem 6.6. $T_{\text{comp}, \oplus}(m, d, Q) = O((Q \log^2(dw))/w + m^2)$ for $m \geq d^{68} \log^8 d$.

Proof. We focus on solving Problem 6.5(i), as (ii) is similar (and easier). We proceed as in the proof of Theorem 5.4. Consider a block β of r consecutive indices in $[m]$. Build the 8-dimensional arrangement \mathcal{A} of $O(rd^4)$ surfaces $\{(x, y, x', y', x'', y'', x''', y''') \in \mathbb{R}^8 : \xi((x, y), B_j[v]^*, (x', y'), B_j[v']^*) = \xi((x'', y''), B_j[v'']^*, (x''', y'''), B_j[v''']^*)\}$ over all $j \in \beta$ and $v, v', v'', v''' \in [d]$. The arrangement

has $O((rd^4)^8) = O(r^8d^{32})$ cells, and we can build a data structure in $O(r^8d^{32}\log^{O(1)}d)$ time supporting point location queries in \mathcal{A} in $O(\log d)$ time [2]. Create a table $f : [m] \times [d]^4 \rightarrow [O(r^8d^{32})]$ that maps (i, u, u', u'', u''') to the index of the cell of \mathcal{A} that contains the point $(A_i[u]_x^*, A_i[u]_y^*, A_i[u']_x^*, A_i[u']_y^*, A_i[u'']_x^*, A_i[u'']_y^*, A_i[u''']_x^*, A_i[u''']_y^*)$. The table can be constructed in $O(md^4 \cdot \log d)$ time by repeated point location queries. Next create a table $g : [O(r^8d^{32})] \times \beta \times [d]^4 \rightarrow \{\text{true}, \text{false}\}$ that maps $(\gamma, j, v, v', v'', v''')$ to whether $\xi((x, y), B_j[v]^*, (x', y'), B_j[v']^*) = \xi((x'', y''), B_j[v'']^*, (x''', y'''), B_j[v''']^*)$ for points $(x, y, x', y', x'', y'', x''', y''')$ in the cell γ of \mathcal{A} . The table can be constructed in $O(r^8d^{32} \cdot rd^4) = O(r^9d^{36})$ time. Map each tuple $(u, v, u', v', u'', v'', u''', v''')$ in Q_{ij} to $(i, j, u, v, u', v', u'', v'', u''', v''')$. Concatenate the Q_{ij} 's over all $(i, j) \in [m] \times \beta$. In the concatenated list, map $(i, j, u, v, u', v', u'', v'', u''', v''')$ to $(f[i, u, u', u''], j, v, v', v'', v''')$, and then to $g[f[i, u, u', u''], j, v, v', v'', v''']$, by two successive applications of Fact 2.1(b) in $O(\sum_i \sum_{j \in \beta} (|Q_{ij}| \log^2(dmw))/w + md^4 + r^9d^{36})$ time. We can then split the resulting list to get the answers to each Q_{ij} for all $(i, j) \in [m] \times \beta$.

Repeating the process for all $O(m/r)$ blocks β gives total time $O((Q \log^2(dmw))/w + (m/r) \cdot (md^4 \log d + r^9d^{36}) + m^2) = O((Q \log^2(dmw))/w + m^2)$ by setting $r = d^4 \log d$, assuming that $m \geq d^{68} \log^8 d$. \square

Next we redefine Problem 3.9 as follows:

Problem 6.7. (Batched $A \oplus B$ Sorting)

- (i) Given lists $A_1, \dots, A_m, B_1, \dots, B_m$ of d long segments each, and given “query” lists Q_{ij} with $|Q_{ij}| \leq (d\ell)^{O(1)}$ and $\sum_{i,j} |Q_{ij}| = Q$, where each element of Q_{ij} is a sequence of ℓ quadruples in $[d]^4$, sort each sequence $\langle (u_1, v_1, u'_1, v'_1), \dots, (u_\ell, v_\ell, u'_\ell, v'_\ell) \rangle$ in Q_{ij} so that at the end, $\xi(A_i[u_1]^*, B_j[v_1]^*, A_i[u'_1]^*, B_j[v'_1]^*) \leq \dots \leq \xi(A_i[u_\ell]^*, B_j[v_\ell]^*, A_i[u'_\ell]^*, B_j[v'_\ell]^*)$, for each $i, j \in [m]$.
- (ii) Given lists $A_1, \dots, A_m, B_1, \dots, B_m$ of d long segments each, and given “query” lists Q_{ij} with $|Q_{ij}| \leq (d\ell)^{O(1)}$ and $\sum_{i,j} |Q_{ij}| = Q$, where each element of Q_{ij} is a sequence containing a real number $x_0 \in \mathbb{R}$ followed by ℓ pairs in $[d]^2$, sort each sequence $\langle x_0, (u_1, v_1), \dots, (u_\ell, v_\ell) \rangle$ in Q_{ij} so that at the end, the lines $\lambda(A_i[u_1]^*, B_j[v_1]^*), \dots, \lambda(A_i[u_\ell]^*, B_j[v_\ell]^*)$ are in increasing y -order at the x -value x_0 , for each $i, j \in [m]$.

Let $T_{\text{sort}, \oplus}(m, d, Q)$ be the time complexity of the above problem.

Theorem 6.8. $T_{\text{sort}, \oplus}(m, d, \ell, Q) = O(((\ell Q \log^2(d\ell w))/w + Q + m^2) \cdot \log^2 \ell)$ for $m \geq \ell^{69/2} d^{68} \log^8(d\ell)$.

Proof. The batched $A \oplus B$ sorting algorithm for Problem 6.7(i) proceeds as before in the proof of Theorem 3.10, using the batched $A \oplus B$ comparison subroutine for Problem 6.5(i) from Theorem 6.6.

Problem 6.7(ii) requires more effort, to resolve the comparisons in each round of the sorting network. Observe that $\lambda(A_i[u]^*, B_j[v]^*)$ is lower than $\lambda(A_i[u']^*, B_j[v']^*)$ at x -value x_0 iff $\xi(A_i[u]^*, B_j[v]^*, A_i[u']^*, B_j[v']^*) \geq x_0$ and $\mu(A_i[u]^*, B_j[v]^*) \leq \mu(A_i[u']^*, B_j[v']^*)$ are both true or both false.

In each round, we thus have to compare $\xi(A_i[u]^*, B_j[v]^*, A_i[u']^*, B_j[v']^*)$ with x_0 , and compare $\mu(A_i[u]^*, B_j[v]^*)$ with $\mu(A_i[u']^*, B_j[v']^*)$, for $O(\ell)$ tuples (u, v, u', v') , for each $t \in [|Q_{ij}|]$ and $i, j \in [m]$. The second type of comparisons can be handled by Problem 6.5(ii). For the first type, we first sort the tuples by $\xi(A_i[u]^*, B_j[v]^*, A_i[u']^*, B_j[v']^*)$ via Problem 6.7(i) (some setup for the bit-packed

input/output is required, but by now these steps should be straightforward, using Fact 2.1(a). Then we can resolve all these comparisons with x_0 by a standard binary search for x_0 in $O(\log \ell)$ time, for each $t \in [|Q_{ij}|]$ and $i, j \in [m]$. The total time for all these binary searches is $O(Q \log \ell)$. With $O(\log \ell)$ rounds of invocation to Problem 6.7(i), the running time increases by another $O(\log \ell)$ factor. \square

6.4 Putting Everything Together via 2-Level Grouping

Lemma 6.9. $T_{\text{search}, \oplus}(m, d, Q) = O((T_{\text{select}, \oplus}(\ell m, d/\ell, \ell Q) + T_{\text{sort}, \oplus}(m, d, \ell, Q) + \ell Q/w + Q + m^2 + \ell^2 m^2/\bar{w}) \cdot \log^{O(1)}(d\bar{w}) + dm \log d)$ for any given $\ell \leq \min\{d, \bar{w}\}$.

Proof. We adapt the proof of Lemma 3.11. Sort each A_i and B_j by the y -values of their endpoints on the edge e_0 , and divide A_i into sublists $A_{i1}, \dots, A_{i\ell}$ of size $d_0 := d/\ell$, and B_j into sublists $B_{j1}, \dots, B_{j\ell}$ of size d_0 . Let α_{ip} denote the region between $A_{ip}[1]$ and $A_{i,p+1}[1]$ within Δ_0 , and let β_{jq} denote the region between $B_{jq}[1]$ and $B_{j,q+1}[1]$ within Δ_0 .

Consider a fixed (i, j) and a fixed $c \in C_{ij}$. For each $c \in C$, there are two types of intersections to count:

1. intersections in $A_{ip} \oplus B_{jq}$ where $\alpha_{ip} \cap \beta_{jq}$ intersects $\partial(c \cap \Delta_0)$, and
2. intersections in $A_{ip} \oplus B_{jq}$ where $\alpha_{ip} \cap \beta_{jq}$ is in the interior of $c \cap \Delta_0$.

We describe an algorithm $\text{search}(c, A_i, B_j)$ to count intersections of type 1.

For type-1 intersections, note that at least one of the four points in $A_{ip}[1] \cap \partial(c \cap \Delta_0)$ and $A_{i,p+1}[1] \cap \partial(c \cap \Delta_0)$ lies in the region β_{jq} , or at least one of the four points in $B_{jq}[1] \cap \partial(c \cap \Delta_0)$ and $B_{j,q+1}[1] \cap \partial(c \cap \Delta_0)$ lies in the region α_{ip} . Without loss of generality, assume that the left point in $A_{ip}[1] \cap \partial(c \cap \Delta_0)$ lies in the region β_{jq} ; each of the other cases can be handled by a similar algorithm (we just have to be careful not to double-count). The algorithm is given by the pseudocode below, and works as follows: For each $p \in [\ell]$, first find the unique index q_p such that the left point in $A_{ip}[1] \cap \partial(c \cap \Delta_0)$ lies in the region β_{jq_p} (i.e., between $B_{jq_p}[1]$ and $B_{j,q_p+1}[1]$), by binary search (lines 2–3). Then find the x -rank k_p of c_x^* among the vertices in $\mathcal{A}((A_{ip} \oplus B_{jq_p})^*)$, by binary search (lines 4–5). Finally, find the y -rank k'_p of c_y^* among the lines in $\mathcal{A}((A_{ip} \oplus B_{jq_p})^*)$ at x -value c_x^* , by another binary search (lines 6–7). Then k'_p tells us how many intersections in $A_{ip} \oplus B_{jq_p}$ lie inside c .

$\text{search}(c, A_i, B_j)$:

1. initialize $q_1 = \dots = q_\ell = k_1 = \dots = k_\ell = k'_1 = \dots = k'_\ell = 1$
2. for $s = \ell/2, \ell/4, \dots, 1$:
3. for each $p \in [\ell]$,
 if the left point in $A_{ip}[1] \cap \partial(c \cap \Delta_0)$ is below $B_{j,q_p+s}[1]$ then $q_p := q_p + s$
4. for $s = d_0^4/2, d_0^4/4, \dots, 1$:
5. for each $p \in [\ell]$,
 if $c_x^* > (x\text{-value of the } (k_p + s)\text{-th leftmost vertex in } \mathcal{A}((A_{ip} \oplus B_{jq_p})^*))$ then $k_p := k_p + s$
6. for $s = d_0^2/2, d_0^2/4, \dots, 1$:
7. for each $p \in [\ell]$,
 if c^* is above the $(k'_p + s)$ -th lowest line at the x -value of the k_p -th leftmost vertex
 in $\mathcal{A}((A_{ip} \oplus B_{jq_p})^*)$ then $k'_p := k'_p + s$
8. for each $p \in [\ell]$, add k'_p to the count

As before, the main idea is to run $\text{search}(c, A_i, B_j)$ simultaneously for all $c \in C_{ij}$ and all $i, j \in [m]$, using bit-packed lists.

To implement line 3, we need to test which side of the segment c contains the point $A_{ip}[1] \cap B_{j, q_p+s}[1]$, i.e., which side of the line $\lambda(A_{ip}[1]^*, B_{j, q_p+s}[1]^*)$ contains the point c^* , for each $p \in [\ell]$. We can first sort these lines in y -order at x -value c_x^* by batched $A \oplus B$ sorting, specifically, Problem 6.7(ii). Then we can resolve all these line comparisons with c^* by standard binary search in y in $O(\log \ell)$ time.

To implement line 5, we apply batched $A \oplus B$ selection, namely, Problem 6.2(i), to obtain the indices to the $(k_p + s)$ -th leftmost vertex in $\mathcal{A}((A_{ip} \oplus B_{jq})^*)$ for each $p \in [\ell]$, and then sort these vertices by x by batched $A \oplus B$ sorting, namely, Problem 6.7(i). Then we can resolve all the comparisons with c_x^* by standard binary search in x in $O(\log d_0)$ time.

To implement line 7, we apply batched $A \oplus B$ selection, namely, Problem 6.2(ii), to obtain the index to the $(k'_p + s)$ -th lowest line at the x -value of the k_p -th leftmost vertex in $\mathcal{A}((A_{ip} \oplus B_{jq})^*)$ for each $p \in [\ell]$, and then sort these lines at x -value c_x^* by batched $A \oplus B$ sorting, namely, Problem 6.7(ii). Then we can resolve all the line comparisons with c^* by standard binary search in x in $O(\log d_0)$ time.

The details of the bit-packed list manipulation and the running-time analysis are as before.

For type-2 intersections, for a fixed (i, j) and fixed $c \in C_{ij}$, it suffices to count, for each p , the number of β_{jq} 's whose intersection with α_{ip} is in the interior of $c \cap \Delta_0$, and then multiply the total number by d^2 . In other words, we want to count the number of β_{jq} 's that are above both the left points in $A_i[1] \cap \partial(c \cap \Delta_0)$ and $A_{i+1}[1] \cap \partial(c \cap \Delta_0)$ and below both the right points in $A_i[1] \cap \partial(c \cap \Delta_0)$ and $A_{i+1}[1] \cap \partial(c \cap \Delta_0)$, or vice versa. The count can be easily computed after finding the index q of the region β_{jq} containing each of the four points in $A_i[1] \cap \partial(c \cap \Delta_0)$ and $A_{i+1}[1] \cap \partial(c \cap \Delta_0)$. Each of these four indices can be found by binary search, in a manner similar to lines 2–3 above. Thus, the running time is similar. \square

Corollary 6.10. *Offline halfplane range searching for bichromatic segment intersections in the long case can be solved in $O((n^2/(w \log n)) \log^{O(1)} w) \leq O((n^2/\log^2 n)(\log \log n)^{O(1)})$ time, assuming that $w = \Omega(\log n)$ and $w \leq n^{o(1)}$.*

6.5 Reduction to the Long Case

Finally, we can solve the problem for arbitrary sets A and B of n disjoint (but not necessarily long) segments and an arbitrary set C of n triangles:

Corollary 6.11. *Offline triangle range searching for bichromatic segment intersections can be solved in $O((n^2/(w \log n)) \log^{O(1)} w) \leq O((n^2/\log^2 n)(\log \log n)^{O(1)})$ time, assuming that $w = \Omega(\log n)$ and $w \leq n^{o(1)}$.*

Proof. We first construct a cutting with $O(r^2)$ cells, each intersecting $O(n/r)$ segments and triangle edges. The cutting and its conflict lists (list of segments and triangle edges intersecting each cell) can be generated in $O(nr)$ time [15]. We further subdivide each cell by vertical lines at each segment endpoint and triangle vertex, and triangulate the cell (and the clipped triangles in C), so that each cell has only long segments and long triangles. The number of cells increases to $O(r^2 + n)$. We then invoke Corollary 6.10 in each cell. The total time is $O((r^2 + n) \cdot ((n/r)^2/(w \log n)) \log^{O(1)} w + nr)$.

We also need to determine whether a triangle completely contains any cell with at least one point in $A \oplus B$. This can be done by n queries in a standard multi-level range searching structure on the $O(r^2 + n)$ cells, in $O((r^2 + n) \cdot ((r^2 + n)n)^{2/3} \log^{O(1)} n)$ time [26].

Setting $r = \sqrt{n}$ yields the result. \square

7 Offline Reverse Range Searching for Bichromatic Segment Intersections

We next adapt the algorithms in Section 6 to solve the geometric problems (v)–(vii) stated in the introduction; the running time is slightly worse. It suffices to consider (vii) offline reverse triangle range searching for bichromatic segment intersections, since the other two problems reduce to it.

As in Section 6, we first concentrate on the special case where all line segments and triangles are long. We may again assume that the long triangles in C are halfplanes, since we can replace each long triangle with its bounding halfplanes, so that the number of long triangles not containing an intersection point $q \in \Delta_0$ is equal to the sum of the number of halfplanes not containing q .

7.1 Reduction to Batched $A \oplus B$ Searching/Selection

Let $T_{\text{search-report},\oplus}(m, d, Q)$ be the time complexity of the variant of Problem 6.1, where “count the number of” is replaced by “report all”. Note that the output for each $c \in C_{ij}$ and $i, j \in [m]$ can be encoded in $O(d \log d)$ bits as a sequence of pairs of the form (u, I_u) , where $u \in [d]$ and I_u is the interval of all v 's such that $A_i[u] \cap B_j[v]$ lies inside c (note that I_u is indeed an interval, assuming that the segments are sorted). The total output size is $O((dQ \log d)/w + Q + m^2)$. (We will not need block representations here.)

Let $T_{\text{select-report},\oplus}(m, d, Q)$ be the time complexity of the variant of Problem 6.1, where “find the k' -th lowest” is replaced by “find the first k' lowest”. We use the same output encoding scheme.

Like before, $T_{\text{search-report},\oplus}(m, d, Q) = O((T_{\text{select-report},\oplus}(m, d, Q) + (dQ \log d)/w + Q + m^2) \cdot \log d)$ (the $(dQ \log d)/w$ term is due to the output size).

Lemma 7.1. *Offline reverse halfplane range searching for bichromatic segment intersections in the long case can be solved in $O(T_{\text{search-report},\oplus}(n/d, d, n^2/d) + (n^2 \log^2 d)/w + n^2/d + n \log n)$ time for any given $d \leq n$.*

Proof. We follow the proof of Lemma 6.3. For each (i, j) , we maintain an array of “local” counters, one for each intersection point in $A_i \oplus B_j$, as well as a “global” counter. The actual count for an intersection point in $A_i \oplus B_j$ is equal to its local counter plus the global counter for (i, j) . (We can easily obtain the minimum/maximum count with this representation scheme.)

For the type-1 intersections, we generate the C_{ij} 's as before and solve the reporting variant of Problem 6.1. Consider a fixed (i, j) . Take a chunk γ of d elements in C_{ij} and sort all the $O(d^2)$ pairs (u, I) of these elements' outputs by u in $O((d^2 \log^2 d)/w + 1)$ time (using Fact 2.1(i)) and split into sublists with a common u . For each sublist, sort the endpoints of its intervals, and perform a linear scan to determine the number of intervals containing each $v \in [d]$; the linear scans take $O((d^2 \log d)/w + d)$ total time and give the number of elements $c \in \gamma$ containing each $A_i[u] \cap B_j[v]$, which we can add to a temporary local counter for $A_i[u] \cap B_j[v]$ in $O((d^2 \log d)/w + 1)$ time, assuming the invariant that the temporary local counters are bounded by $d^{O(1)}$. After seeing d chunks, we add the temporary local counters to the current local counters in $O(d^2)$ time and reset the temporary local counters (to ensure the invariant). The total time for these steps is $O(\sum_{i,j} (|C_{ij}|/d + 1)((d^2 \log^2 d)/w + d) + \sum_{i,j} (|C_{ij}|/d^2)d^2) = O((n^2 \log^2 d)/w + n^2/d)$, since $\sum_{i,j} |C_{ij}| = O(n^2/d)$ and there are $O(n^2/d^2)$ terms.

For type-2 intersections, for each (i, j) , we compute the number of $c \in C$ such that $\alpha_i \cap \beta_j$ is strictly inside $c \cap \Delta_0$, and add the number to the global counter. These numbers can be computed in $O(n^2/d)$ total time, since the number for (i, j) can be computed from that for $(i, j - 1)$ by examining the halfplanes in C_{ij} and $C_{i, j-1}$. \square

7.2 Batched $A \oplus B$ Selection via Cuttings in Near-Logarithmic Dimensions

Theorem 7.2. $T_{\text{select-report}, \oplus}(m, d, Q) = O((dQ \log d)/w + Q + m^2)$ for $d \leq \delta \log m / \log \log m$ for a sufficiently small constant $\delta > 0$.

Proof. As in the proof of Theorem 6.4 (except that it is sufficient to follow the simpler Theorem 3.5 instead of Theorem 3.6). The $(dQ \log d)/w$ term is due to the output size. \square

It follows that $T_{\text{search-report}, \oplus}(m, d, Q) = O(((dQ \log d)/w + Q + m^2) \cdot \log d) = O((Q + m^2)(\log \log m)^{O(1)})$ for $d = \delta \log m / \log \log m$, and so by Lemma 7.1, we obtain:

Corollary 7.3. *Offline reverse halfplane range searching for bichromatic segment intersections in the long case can be solved in $O((n^2 / \log n)(\log \log n)^{O(1)})$ time.*

We think some small improvements are possible with a more complicated algorithm, incorporating the ideas in Sections 6.3 and 6.4, but currently we have not yet been able to achieve $O((n^2 / \log^2 n)(\log \log n)^{O(1)})$ running time for this problem.

7.3 Reduction to the Long Case

Corollary 7.4. *Offline reverse triangle range searching for bichromatic segment intersections can be solved in $O((n^2 / \log n)(\log \log n)^{O(1)})$ time.*

Proof. The reduction to the long case is as in the proof of Corollary 6.11. The only main change is that for each cell in the cutting, we need a global counter for the number of triangles completely containing the cell. This can be done by $O(r^2 + n)$ queries in a standard multi-level range searching structure on $O(n)$ triangles, in $O((n + r^2 + ((n + r^2)r^2)^{2/3} \log^{O(1)} n))$ time [26]. \square

8 Final Remarks

The analogy between 3SUM and (select,+)-convolution suggests how close in hindsight Bremner et al.'s method for (select,+)-convolution [7] was to an $O((n^2 / \log n)(\log \log n)^{O(1)})$ -time algorithm for 3SUM. Their method was even closer to solving the *convolution-3SUM* problem: given three lists A , B , and C of n real numbers, decide whether there exist $i, u \in [n]$ with $C[i] = A[u] + B[i - u]$. (Bremner et al. basically solved the batched $A \hat{+} B$ selection problem with group size d near $\log n$, but convolution-3SUM reduces to batched $A \hat{+} B$ selection by simultaneous binary searches, with an extra $O(\log d) = O(\log \log n)$ factor.)

Note that (select,+)-convolution is *integer-3SUM-hard*, since the integer version of 3SUM reduces to the integer version of convolution-3SUM [28, 25] (with some extra logarithmic factor overhead), which in turn reduces to the integer version of (select,+)-convolution by simultaneous binary searches (with at most another logarithmic factor overhead).

For the geometric problems considered here, we can obtain a truly subquadratic $O(n^{2-\varepsilon})$ upper bound on the decision tree complexity, using just a subset of the ideas in Section 6. (For instance,

we can just set $w \approx n^\varepsilon$ for a sufficiently small constant ε when working in the decision tree model, since the cost of bit manipulation is not counted; as logarithmic factors are less important here, we do not even need the part concerning cuttings in near-logarithmic dimensions.) To optimize ε in the exponent, the approach of Barba et al. [6] might be better. It is unclear if the recent techniques of Kane, Lovett, and Moran [24] could be applicable to these geometric problems (since these problems require point location in high-dimensional arrangements of *nonlinear* surfaces, whereas Kane et al.’s technique deals with point location in arrangements of hyperplanes with integer coefficients).

We hope that our techniques will find further applications in computational geometry. Currently, the techniques are limited to geometric settings where the objects in each of the input sets A and B are disjoint. In particular, they do not work when A and B are arbitrary lines in \mathbb{R}^2 . It remains open whether there is a subquadratic algorithm for the degeneracy testing for n lines in \mathbb{R}^2 (i.e., deciding whether there exist 3 lines meeting at a common point), or equivalently, by duality, degeneracy testing for n points in \mathbb{R}^2 (i.e., deciding whether there exist 3 collinear points). Since many 3SUM-hard problems in computational geometry are actually “collinear-triple-hard”, such problems remain unaffected by our techniques at present, unfortunately.

The author’s latest combinatorial algorithm for APSP or $(\min,+)$ -matrix multiplication [13] runs in $O((n^3/\log^3 n)(\log \log n)^{O(1)})$ time (building on earlier combinatorial algorithms for Boolean matrix multiplication [12, 30]), but the ideas there do not seem applicable to 3SUM or $(\text{select},+)$ -matrix multiplication to shave off a third logarithmic factor. Williams [29] gave a breakthrough non-combinatorial $n^3/2^{\Omega(\sqrt{\log n})}$ -time algorithm for APSP or $(\min,+)$ -matrix multiplication, using the polynomial method, but these ideas have so far failed to give better results for 3SUM or $(\text{select},+)$ -matrix multiplication. We do not know how to obtain still faster algorithms for the integer special case of the 3SUM problem, although truly subquadratic algorithms have been found in some interesting special cases [14].

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