Drawing $K_{2,n}$: A Lower Bound*

Therese Biedl[†] Timothy M. Chan[†] Alejandro López-Ortiz[†]

1 Introduction

In graph drawing [1], the main objective is to obtain a representation of a graph in the plane under some aesthetic or functional criteria. For the purposes of visualization and chip layout, one would like to devise planar embeddings in a rectangular grid that have both small area and small aspect ratio. Here, the area of an embedding is defined as WH and the aspect ratio as $\max\{W/H,\ H/W\}$, if the minimum enclosing axis-parallel rectangle has width W and height H.

In this paper, we study *straight-line* drawings, where vertices are mapped to grid points and edges are mapped to noncrossing straight line segments. It is well-known that every planar graph on n vertices has a planar straight-line drawing in an $O(n)\times O(n)$-grid (see for example [4, 5]), hence with $O(n^2)$ area and constant aspect ratio. Also, there are planar graphs on n vertices that require $\Omega(n^2)$ area for any planar embedding [3].

Some planar graphs clearly can be drawn with less than $O(n^2)$ area. An interesting question is therefore whether such graphs can also be drawn with less than $O(n^2)$ area *and* with a constant aspect ratio. In particular, Steve Wismath at the 2001 Graph Drawing Symposium [2] conjectured that this is not possible for a graph containing $K_{2,n}$, which can clearly be drawn in $O(n)$ area. In this note, we show that this is indeed the case: no drawing of $K_{2,n}$ with $O(n)$ area has constant aspect ratio.

2 Preliminaries

$K_{2,n}$ is the complete bipartite graph with two vertices (say a and b) in one class and n vertices (say $v_1, \ldots, v_n$) in the other class, and all possible edges between them. Figure 1 shows an embedding of $K_{2,n}$ in an $n\times 2$ grid. Since clearly $K_{2,n}$ needs $\Omega(n)$ grid points, this drawing has asymptotically optimal area. However, the aspect ratio is $\Theta(n)$. A natural question is hence: What is the smallest area of a drawing of $K_{2,n}$ that has a constant aspect ratio? Or more specifically, is it possible to draw $K_{2,n}$ in an $O(\sqrt{n})\times O(\sqrt{n})$-grid? (It is not difficult to see that the graph $K_{1,n}$ can be drawn that way; in contrast, $K_{3,n}$ is not planar for $n \geq 3$.)

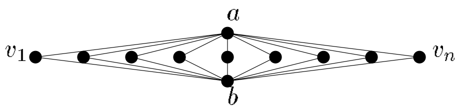

Figure 1: A drawing of $K_{2,n}$ of area $O(n)$.

We first state an easy lemma for counting grid points inside rectangles (one possible self-contained proof is given for completeness):

Lemma 2.1 *A $w\times h$ rectangle (of arbitrary slopes) can contain at most $O((w+1)(h+1))$ grid points.*

Proof: Suppose there are k grid points inside the rectangle. Draw disks of radius 1/2 around each such point. These disks are disjoint and their union has area $(\pi/4)k$, but since the union is contained in a $(w+1)\times(h+1)$ rectangle, $(\pi/4)k \leq (w+1)(h+1)$. □

3 The Proof

Consider a planar straight-line drawing of $K_{2,n}$ in a $W\times H$ grid. We will upper-bound the number of vertices n in terms of W and H, thereby showing that W or H must be large. Without loss of generality, assume $W \geq H$.

Let R be the minimum axis-parallel rectangle enclosing a and b. Let ℓ be the line through a and b, denote by L the distance between a and b, and denote by D the distance of ℓ to an opposite corner of R. Observe that $L \in O(W)$ and $D \in O(H)$. In the sequel, we will only count vertices v_j's in the upper half-plane of ℓ, as vertices in the lower half-plane can be dealt with similarly. (See Figure 2.)

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[†] School of Computer Science, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1, email: {biedl,tmchan,alopez-o}@uwaterloo.ca

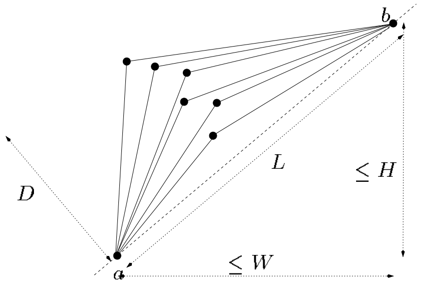

Figure 2: Definition of L and D.

Any horizontal line above R can contain at most one vertex by planarity (see Figure 3), and similarly, any vertical line left of R can contain at most one vertex. Thus, at most $W + H \in O(W)$ vertices can be drawn outside R. We therefore focus our attention now on counting vertices inside R.

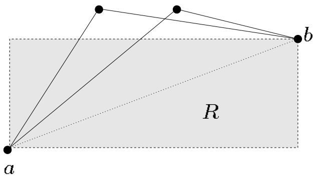

Figure 3: Each row outside R cannot contain two vertices without introducing crossings.

Form $\lceil D \rceil$ strips $S_1, \dots, S_{\lceil D \rceil}$ of width 1, where S_i contains all points above ℓ whose distance to ℓ lies in the interval $(i-1, i]$ (see Figure 4).

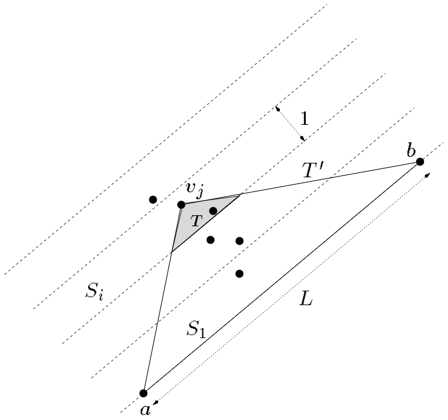

Figure 4: Proof of the claim.

Claim 1 *Strip S_i contains at most $O(L/i+1)$ vertices inside R.*

Proof: Assume that $S_i \cap R$ contains at least one vertex, and let v_j be the farthest such vertex from the line ℓ. Let T' be the triangle $\triangle a v_j b$ and let T be the intersection of T' with S_i (the triangle T is shaded in Figure 4).

Note that no vertex within S_i can be outside T, because otherwise its line towards a or b would cross an edge of T'. So the number of vertices within S_i is bounded by the number of grid points inside T. The height of T is at most 1, and the width of T is at most L/i, because T is similar to T' (which has width L) and has at most $1/i$ times its height. Since $v_j \in R$, the triangle T is obtuse and is therefore contained in a $1 \times (L/i)$ rectangle. By Lemma 2.1, the number of grid points inside T is $O(L/i+1)$, which proves the claim. □

The number of vertices inside R, in all strips together, is at most a constant times

$$\sum_{i=1}^{\lceil D \rceil} \left(\frac{L}{i} + 1 \right) \leq L(1+\ln \lceil D \rceil) + \lceil D \rceil \in O(W \log H).$$

We conclude that the overall number of vertices n is bounded by $O(W \log H)$.

Theorem 1 *Every planar straight-line drawing of $K_{2,n}$ in a $W \times H$ grid with $W \geq H$ satisfies $W \log H \in \Omega(n)$.*

Using this theorem, we can now obtain the desired results.

Corollary 2 *Every planar straight-line drawing of $K_{2,n}$ in a $W \times H$ grid satisfies $\max\{W, H\} \in \Omega(n/\log n)$.*

Proof: Assume $W \geq H$. If $H > n$, then nothing is to show. Otherwise, by Theorem 1 we have $W \in \Omega(n/\log H)$, which is in $\Omega(n/\log n)$ by $H \leq n$. □

Corollary 3 *Every planar straight-line drawing of $K_{2,n}$ with aspect ratio $O(1)$ has area at least $\Omega(n^2/\log^2 n)$.*

Proof: Assume that drawing is in a $W \times H$ grid, with $W \geq H$. By the previous corollary, $W \in \Omega(n/\log n)$. Since the drawing has aspect ratio $O(1)$, we have $H \in \Omega(n/\log n)$ as well, and $WH \in \Omega(n^2/\log^2 n)$. □

Corollary 4 *Every planar straight-line drawing of $K_{2,n}$ with $O(n)$ area has aspect ratio $\Omega(n)$.*

Proof: Assume that drawing is in a $W \times H$ grid, with $W \geq H$. By assumption, $WH \in \Theta(n)$. By Theorem 1, we have $W \log H \in \Omega(n)$. Therefore $H/\log H \in O(1)$, which holds only if $H \in \Theta(1)$, and therefore $W \in \Theta(n)$. □

4 Open problems

The main arguments in our proof (modulo the simple counting trick for vertices outside R) actually hold for any embedding where every pair of vertices is of distance at least 1. It is interesting to see if properties of the integer lattice would allow us to eliminate the extra $\log H$ factor in Theorem 1.

Another question that remains open is if the lower bounds shown hold for planar *polyline* drawings, where edges may bend at grid points.

References

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