On the Bichromatic k-Set Problem

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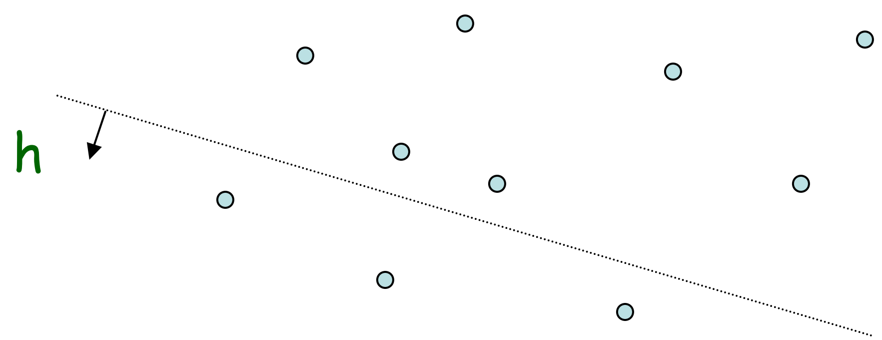
A STANDARD PROBLEM IN COMBINATORIAL GEOMETRY…

The k-Set Problem (in 2D)

- Given n pts P in R^2, a k-set is a subset $P \cap h$ of size k for some halfplane h



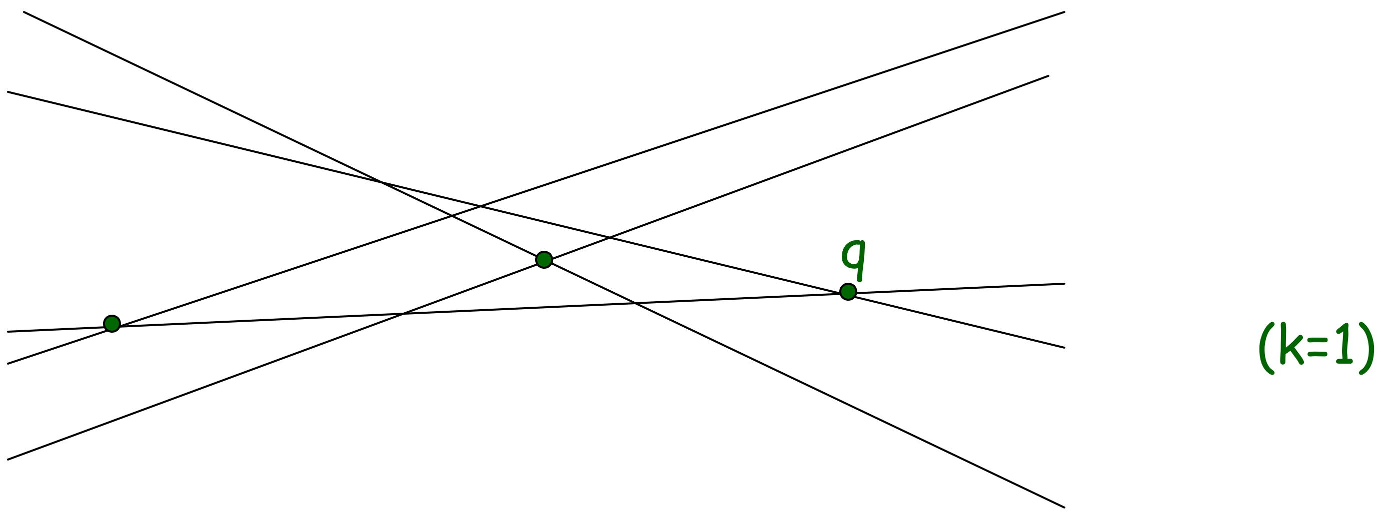
(k=3)

- How many k-sets possible as fn of n & k?

(Note: k=n/2 case called halving line problem)

Equiv. Dual Version: k-Level of Lines

- Given n lines in R^2,
  the level of a pt q := # of lines below q



- How many vertices of level k?

Previous Work: Upper Bds

- Lovász'71 $O(n^{3/2})$
- Erdős,Lovász,Simmons,Straus'73 $O(nk^{1/2})$

  (also by Gusfield'79, Edelsbrunner,Welzl'85)
- Pach,Steiger,Szemerédi [FOCS'89] $O(nk^{1/2}/\log^* n)$
- Dey [FOCS'97] $O(nk^{1/3})$
- C. [FOCS'03] $O(nk^{1/2})$

  (a different proof)

Previous Work: Lower Bds

- Erdős,Lovász,Simmons,Straus'73 $\Omega(n \log k)$
- Klawe,Paterson,Pippenger'82 $n2^{\Omega(\sqrt{\log k})}$

  (for pseudolines)
- Tóth [SoCG'00] $n2^{\Omega(\sqrt{\log k})}$

- Conjecture by Erdős et al.'73:

  $o(n^{1+\varepsilon})$??

New Result (for the Standard 2D Problem)

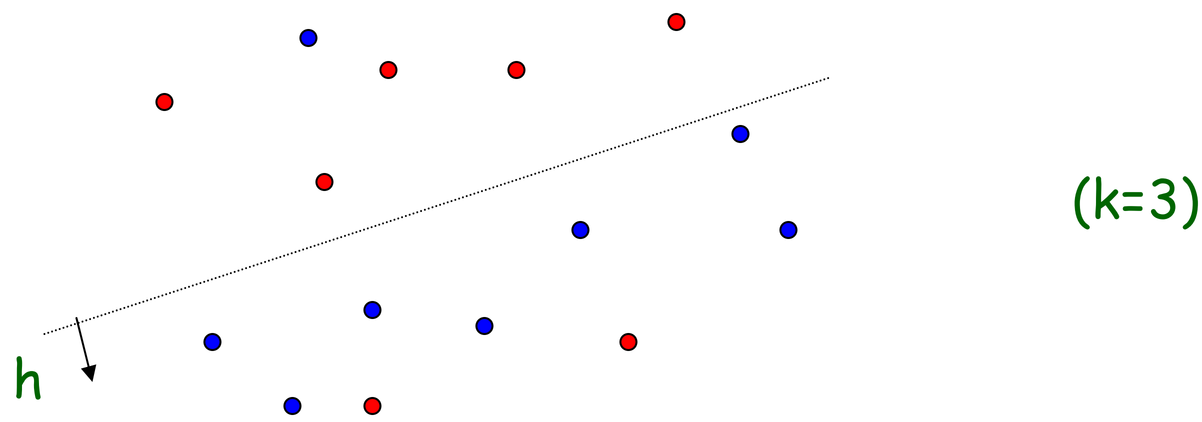
New Result (for the Standard 2D Problem)

- Nothing (sorry!)

A “NEW” VERSION OF THE PROBLEM…

The Bichromatic k-Set Problem (in 2D)

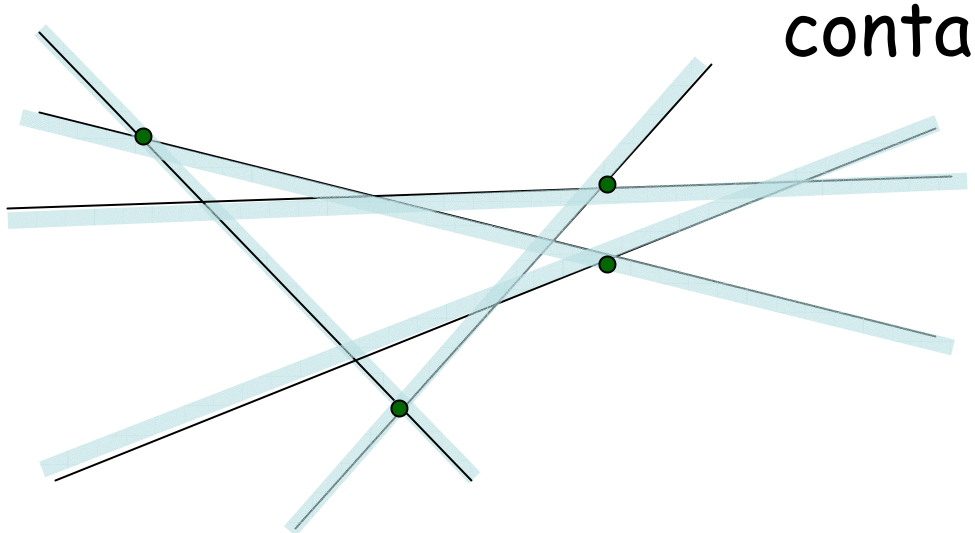
- Given n red pts R & blue pts B in R^2, a k-set is a subset $(R \cap h) \cup (B - h)$ of size k for some halfplane h



- How many k-sets possible as fn of n & k?

Equiv. Version: k-Level of Halfplanes

- Given n halfplanes in R^2,
  the level of a pt q := # of halfplanes not containing q



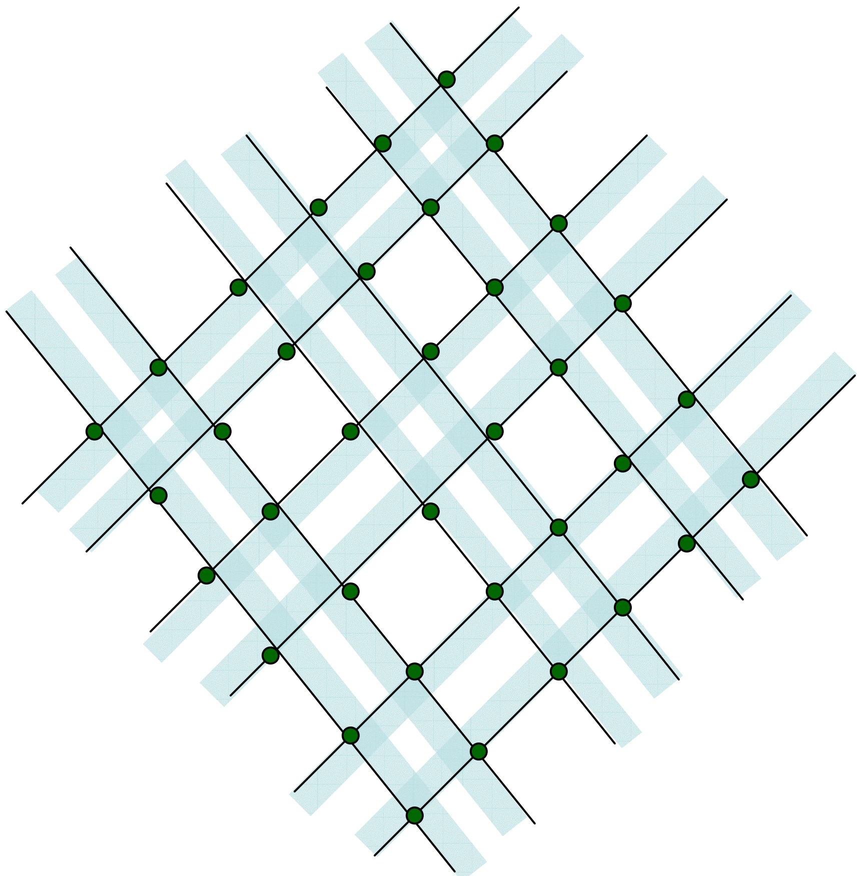
(k=1)

- How many vertices of level k?

(Appl'ns: optimization with k violated constraints…)

Unfortunately…

- subquadratic in n is not possible!



(k ~ n/2)

One "Easy" Upper Bd

- O(nk) [e.g. Clarkson,Shor]

One Previous Work

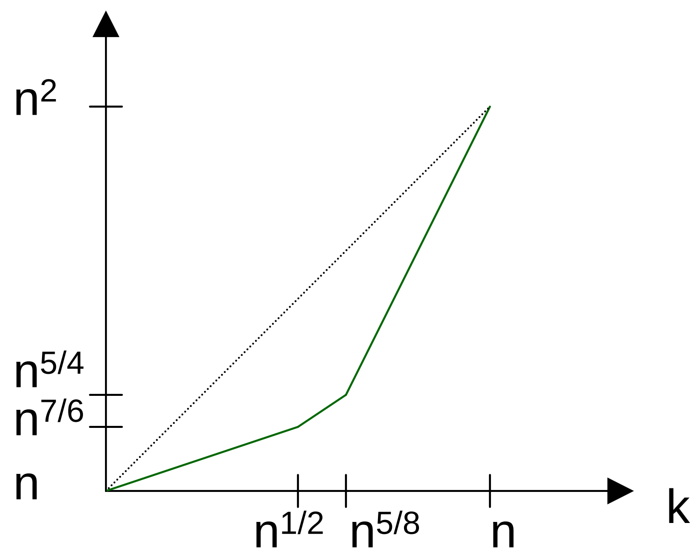
- Linhart'93: improve consts
- Conjecture by Linhart:
  O(nk) is tight

New Result

New Result

- Linhart is wrong (sorry!)

- $O^*(nk^{1/3} + n^{5/6}k^{2/3} + k^2)$



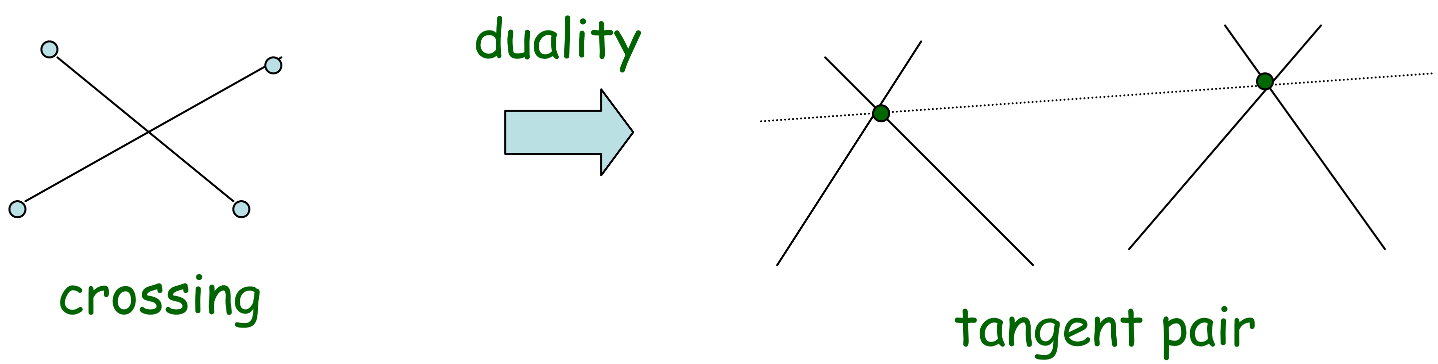
(Note: * hides n^{ε} factors)

Rest of Talk

- Rough sketch of the approach
  - Extend Dey's $O(nk^{1/3})$ proof
  - Extend C.'s $O(nk^{1/2})$ proof
  - Combine…
- Results beyond 2D

Dey's Proof (for the Standard Problem) [FOCS'97]

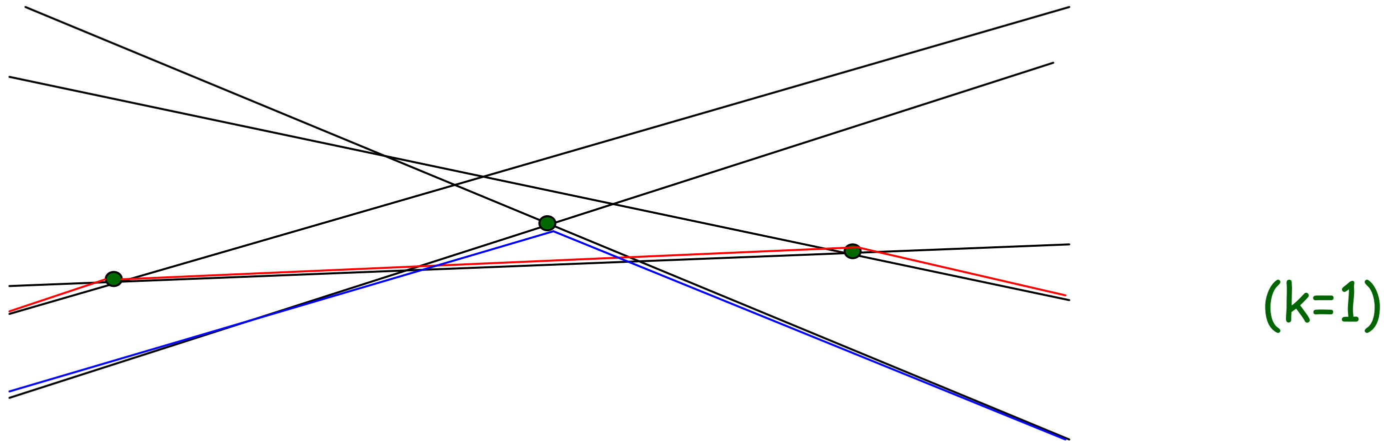
- Crossing Lemma: A geometric graph with n vertices & X crossings has $O(n + n^{2/3}X^{1/3})$ edges
- Equivalently: A set of vertices in an arrangement of n lines with X tangent pairs has cardinality $O(n + n^{2/3}X^{1/3})$



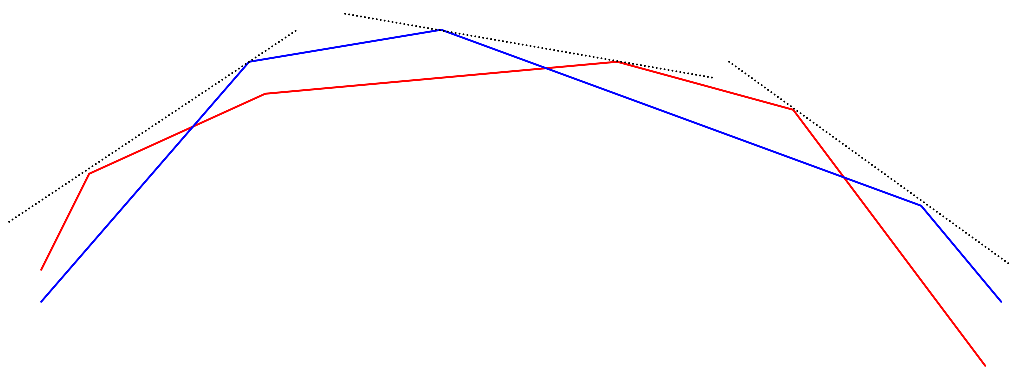
- To show: $X = O(nk)$

Dey's Pf (Cont'd)

- Form k+1 concave chains to visit all k-level vertices



- # tangents betwn 2 chains ≤ # intersect'ns + O(1)



⇒ X ≤ total # intersect'ns + $O(k^2)$ = $O(nk)$

New Proof 1

- Idea: modify Dey's pf

- This time, # concave/convex chains = $O(k + M)$
  where M = # k-level local x-minima/maxima

$\Rightarrow$ Total # intersect'ns = $O(nk + M^2)$

$\Rightarrow$ $X = O(nk + M^2)$

$\Rightarrow$ $O(n + n^{2/3}X^{1/3}) = O(nk^{1/3} + n^{2/3}M^{2/3})$

- Known: $M = O(k^2)$ [Clarkson,Shor/Mulmuley'93]

$\Rightarrow$ $O(nk^{1/3} + n^{2/3}k^{4/3})$

Refined Proof 1

- Idea: divide&conquer [Agarwal,Aronov,C.,Sharir'98]
- Divide into $O(n/k)$ slabs each with $O(k)$ relevant lines

$\Rightarrow O(\sum_i (n_i k^{1/3} + n_i^{2/3} M_i^{1/3}))$

$= O(n/k \cdot [k^{4/3} + k^{2/3} (k^2/(n/k))^{1/3}])$

$= \boxed{O(nk^{1/3} + n^{1/3}k^{5/3})}$

C.'s Proof (for the Standard Problem) [FOCS'03]

- Let t_i = # vertices at level k-i, ..., k+i

- To show: $t_i \lesssim 2i\,(t_{i+1} - t_i)$

$\Rightarrow t_0 = O(t_k / k^{1/2}) = O(nk^{1/2})$

Proof 2

- Idea: modify C.'s pf
- $O^*(nk^{1/2} + M) = O^*(nk^{1/2} + k^2)$

Combining Proof 1 & 2

- $O^*(nk^{1/3} + n^{5/6}k^{2/3} + k^2)$

BEYOND 2D...

The Standard Problem in 3D: Previous Work

- Bárány,Füredi,Lovász [SoCG'89] $O(n^{2.998})$
- Aronov,Chazelle,Edelsbrunner, Guibas,Sharir,Wenger [SoCG'90] $O(n^{8/3}\log^{5/3} n)$
- Eppstein'93 $O(n^{8/3}\log^{2/3} n)$
- Dey,Edelsbrunner [SoCG'93] $O(n^{8/3})$
- Agarwal,Aronov,C.,Sharir'98 $O(nk^{5/3})$
- Sharir,Smorodinsky,Tardos [SoCG'00] $O(nk^{3/2})$
- C. [SODA'05] $O(nk^{1.997})$
- Tóth [SoCG'00] $nk2^{\Omega(\sqrt{\log k})}$

The Bichromatic Problem in 3D: New Result

- $O(nk^{3/2} + n^{0.5034}k^{2.4932} + k^3)$

- Proof approach
  - Extend Sharir,Smorodinsky,Tardos' $O(nk^{3/2})$ proof
  - Extend C.'s $O(nk^{1.997})$ proof
  - Combine…

The Standard Problem in 4D: Previous Work

- Bárány,Füredi,Lovász [SoCG'89] + Živaljević,Vrecića'92 — $O(n^{3.99997})$
- Agarwal,Aronov,C.,Sharir'98 — $O(n^2k^{1.99997})$
- Matoušek,Sharir,Smorodinsky, Wagner'06 — $O(n^2k^{88/45})$

The Bichromatic Problem in 4D: New Result

- $O(n^2k^{3/2} + n^{1.5034}k^{2.4932} + nk^3)$

The Bichromatic Problem in 4D: New Result

- $O(n^2k^{3/2} + n^{1.5034}k^{2.4932} + nk^3)$

Note: new even for the standard k-set problem in 4D !!

(beats $O(n^2k^{88/45})$ for $k \ll n^{0.92}$)

Final Remarks

- New results also for k-level of curved objects, e.g., balls in 2D/3D…
- Open:
  - Improvements in 2D & 3D ??
  - 5D, 6D, … ??
  - More on k-sets for small k ??