Cuttings in 2D Revisited

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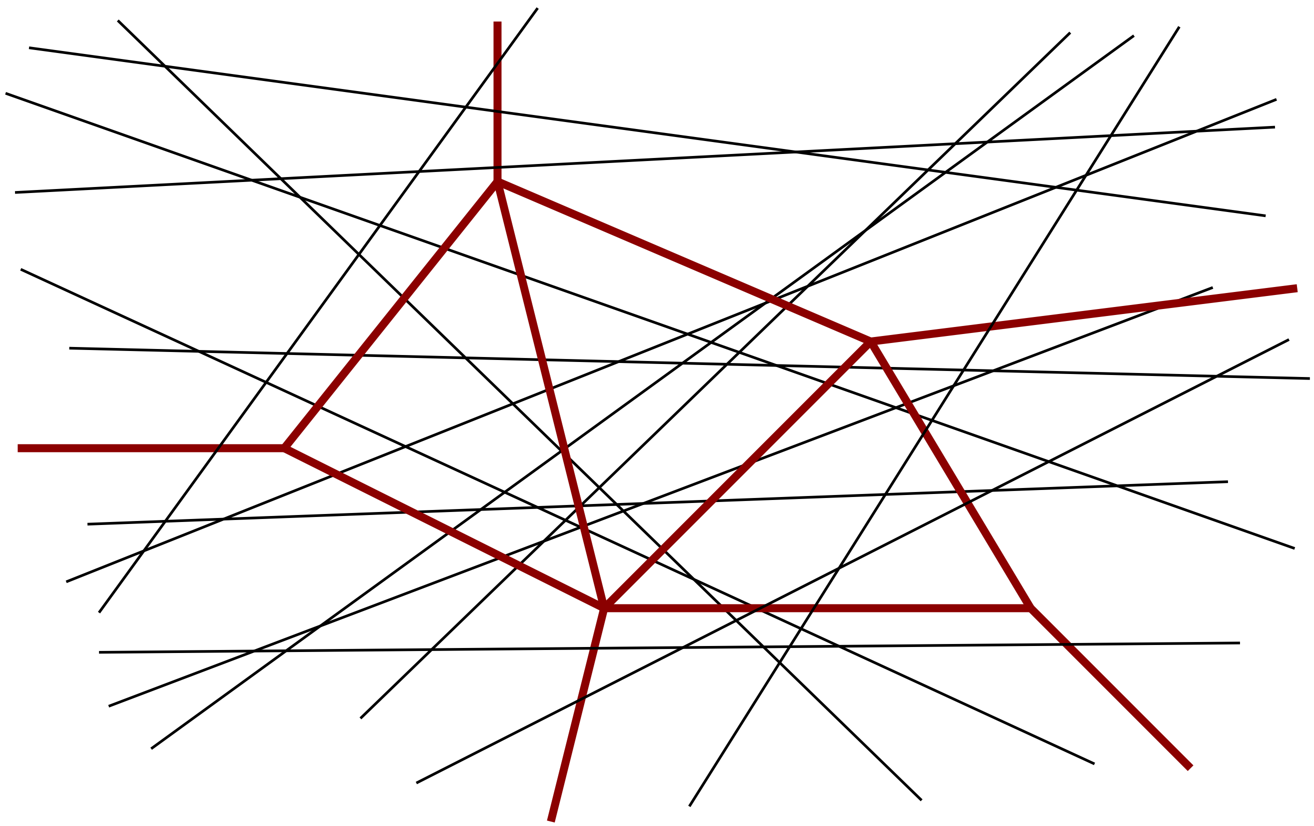
Disclaimers

- theory talk (it's about derandomization!)
- no new result
- a "new" alg'm that isn't totally original. . .

[but I hope it will be "educational". . .
new alg'm fits in 1 slide. . . & no probabilities!]

The Problem

Def'n: Given set L of n lines in 2D,
a $(1/r)$-cutting K is a subdivision into cells
s.t. each cell Δ intersects $\leq n/r$ lines



The Problem

Def'n: Given set L of n lines in 2D,
a $(1/r)$-cutting K is a subdivision into cells
s.t. each cell Δ intersects $\leq n/r$ lines

Remarks:

- cells could be arbitrary/convex/triangles/trapezoids...
- size of K = # cells in K
- conflict list $L_\Delta = \{$all lines in L intersecting $\Delta\}$
- generalizes medians & quantiles in 1D
  ($\exists$ $(1/r)$-cutting of size r in 1D)

The Result

Theorem: In 2D, $\exists$ $(1/r)$-cutting of size $\boxed{O(r^2)}$

It (& its conflict lists) can be computed in time $\boxed{O(nr)}$

Remarks:

- size $O(r^2)$ is optimal
- time $O(nr)$ is optimal if conflict lists are required
  (output size is $\Omega(r^2 \cdot n/r)$)
- In higher D: size $O(r^d)$, time $O(nr^{d-1})$

Why Fundamental

- prune&search in CG
- divide&conquer in CG
- basic tool in range searching
- many applications. . .

An Example Application:

Offline 2D Halfplane Range Counting ("Hopcroft's Problem")

- given n lines & m pts,
  count # pairs (p, ℓ) where pt p is below line ℓ

Naive Sol'n:

$$T(n, m) \leq O(n^2 + m \log n)$$

by constructing arrangement of n lines +
m point location queries

An Example Application:

Offline 2D Halfplane Range Counting ("Hopcroft's Problem")

Fastest Sol'n Known (Almost):

$$T(n,n) \le O(r^2)\,T(n/r, n/r^2) + O(nr) \quad \text{by cutting}$$

$$\le O(r^2)\,T(n/r^2, n/r) + O(nr) \quad \text{by duality}$$

$$\le O(r^2)\left[(n/r^2)^2 + (n/r)\log n\right] + O(nr)$$

by naive sol'n

$$= O(n^2/r^2 + nr\log n)$$

$$= \boxed{O(n^{4/3}\log^{2/3} n)} \quad \text{by setting } r = \left(\tfrac{n}{\log n}\right)^{1/3}$$

[Matoušek'92: log factor improvable to iterated log. . .]

Rest of Talk

I. History

II. “New” Alg’m

III. Coda

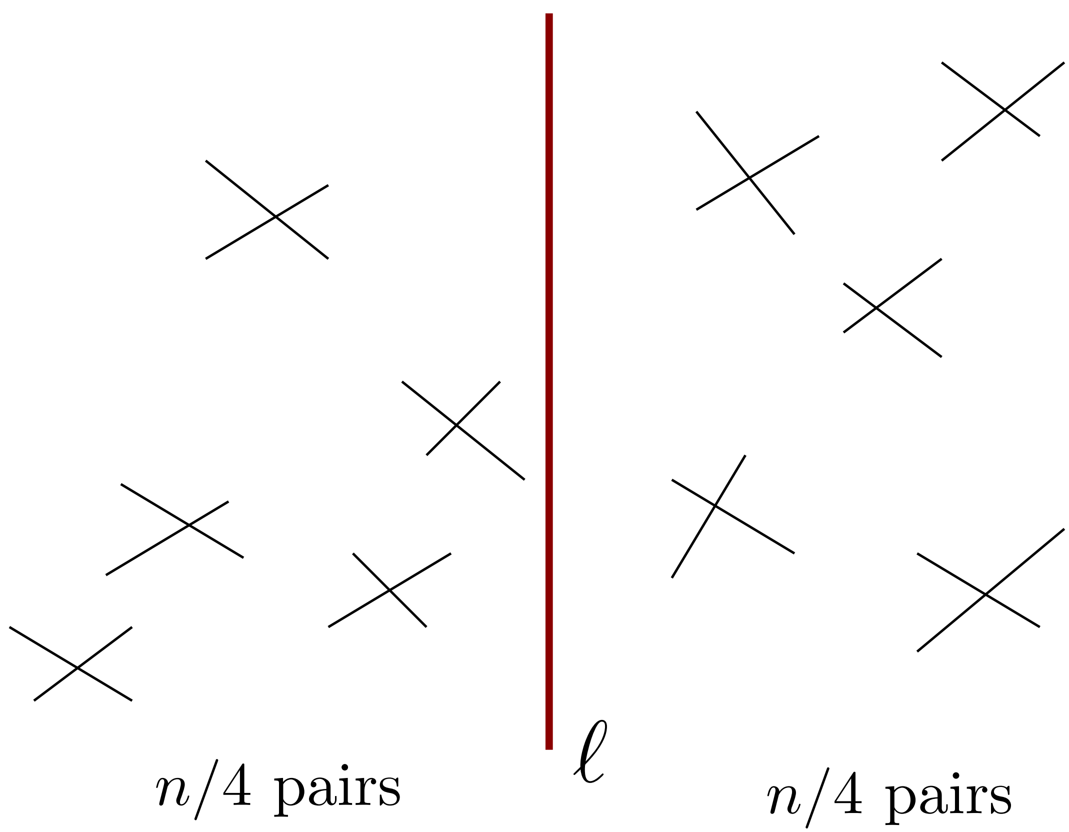
Megiddo, “Linear time algorithm for linear programming in R^3 and related problems”, FOCS’82

Dyer, “Linear time algorithm for two- and three-variable linear programs”, ’84

- $\boxed{(7/8)}$-cutting of size $\boxed{4}$ in linear time

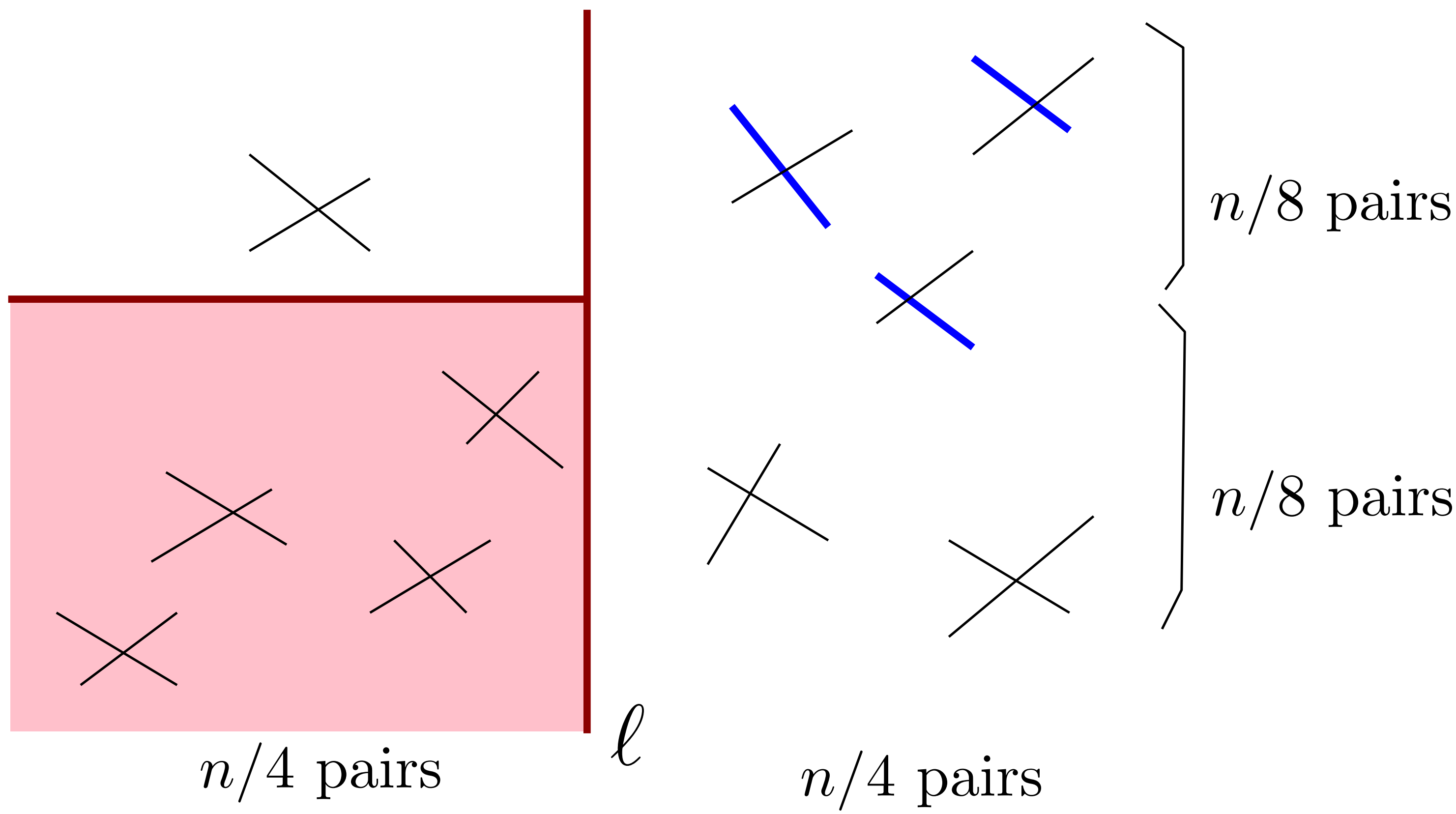
Megiddo/Dyer:
The "How-many-times-can-you-take-medians" Method

1. $m =$ median slope
2. pair lines of slope $< m$ w. lines of slope $> m$
   intersect each pair
   draw median vertical line ℓ thru intersection pts



Megiddo/Dyer:
The “How-many-times-can-you-take-medians” Method

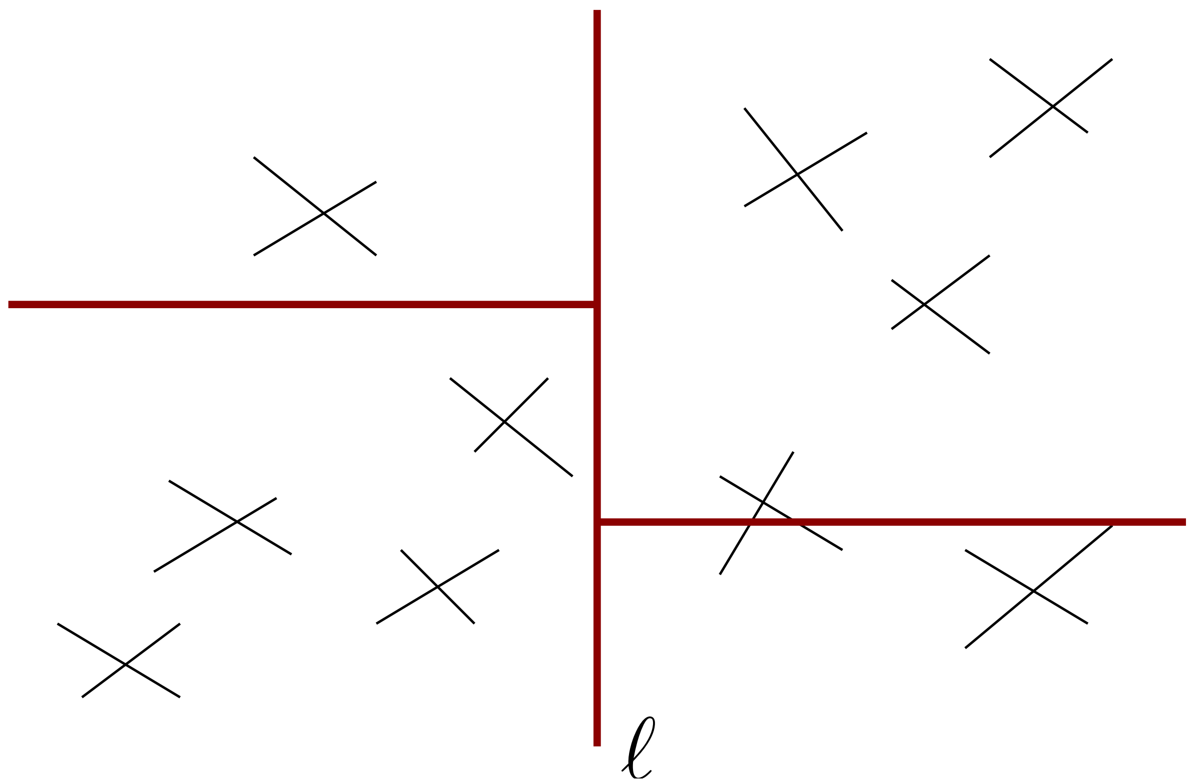
3. on left side of ℓ:
   draw median slope-m line at
   those intersection pts to the right of ℓ



Megiddo/Dyer:

The "How-many-times-can-you-take-medians" Method

3. on left side of ℓ:
   draw median slope-m line at
   those intersection pts to the right of ℓ
4. on right side of ℓ: similar



Megiddo/Dyer:

The "How-many-times-can-you-take-medians" Method

Remarks:

- extends to higher D, but w. horrible consts
- improvable to $(3/4)$-cutting of size 4 [Yamamoto et al.'88]
- can get $(1/r)$-cutting for any r by straightforward recursion:

size $S(r) = 4\,S(\frac{7}{8}r) \;\Rightarrow\; O(r^{\log_{8/7} 4}) = \boxed{O(r^{10.4})}$

time $T(n,r) = 4\,T(\frac{7}{8}n, \frac{7}{8}r) + O(n) \;\Rightarrow\; \boxed{O(nr^{9.4})}$

[alternative: instead of pairing, divide into groups of $r\ldots$]

Clarkson, “A probabilistic algorithm for the post office problem”, STOC’85

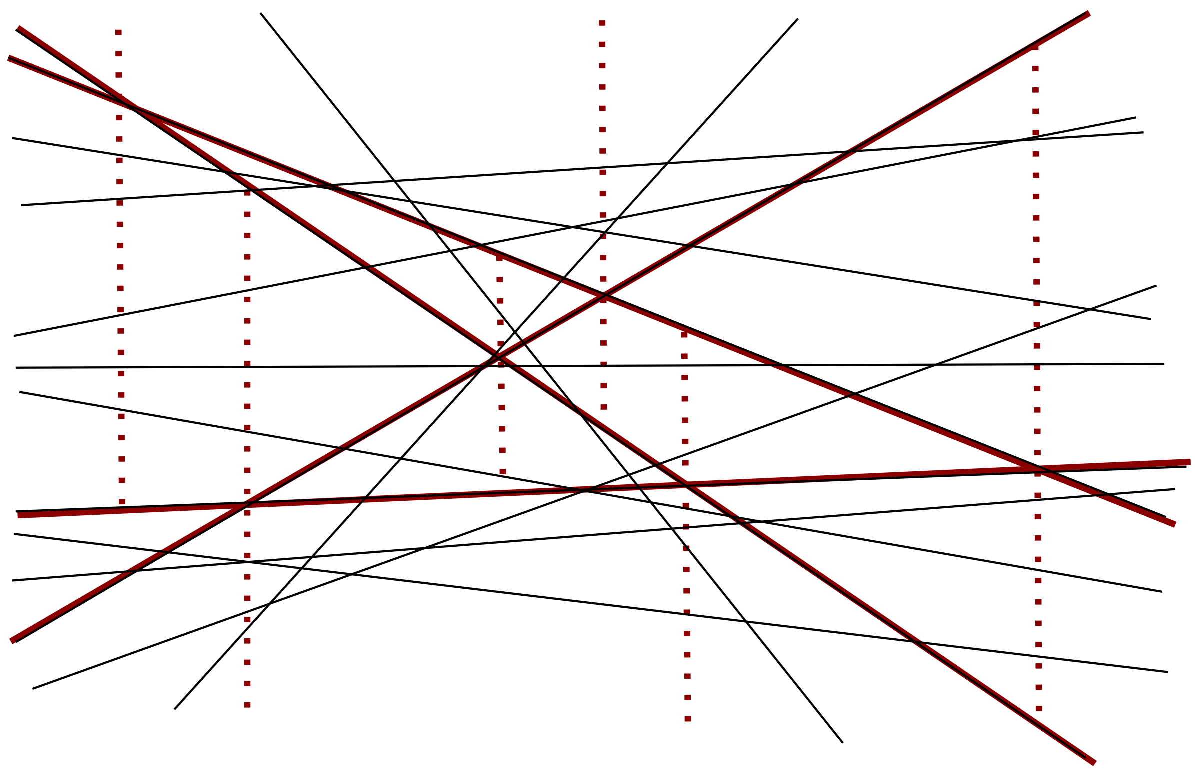
Clarkson, “Further applications of random sampling to computational geometry”, STOC’86

Haussler & Welzl, “Epsilon-nets and simplex range queries”, SoCG’86

- $(1/r)$-cutting of size $\boxed{O((r\log r)^2)}$

Clarkson/Haussler&Welzl: "The Sampling Method"

1. take random sample of r lines
2. return its trapezoidal decomposition (VERY simple!)



Clarkson/Haussler&Welzl: "The Sampling Method"

1. take random sample of r lines
2. return its trapezoidal decomposition (VERY simple!)

- size $O(r^2)$
- each cell intersects $O((n/r)\log r)$ lines w. high probability (analysis omitted)

Clarkson/Haussler&Welzl: "The Sampling Method"

Remarks:

- Chazelle&Friedman'88 removed extra log $\Rightarrow$ first existence proof w. optimal size
- general, extends to higher D
- deterministic alg'ms?

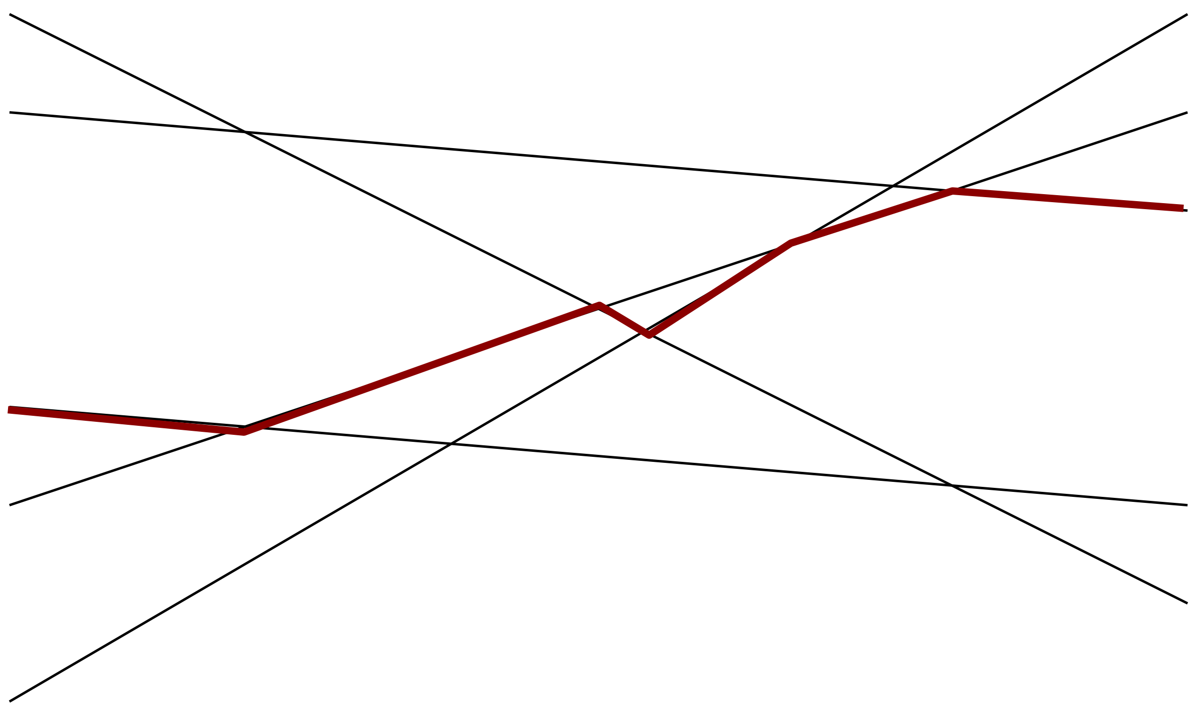
Matoušek, “Construction of epsilon nets”, SoCG’89

- $(1/r)$-cutting of size $\boxed{O(r^2)}$

Matoušek: "The Level Method"

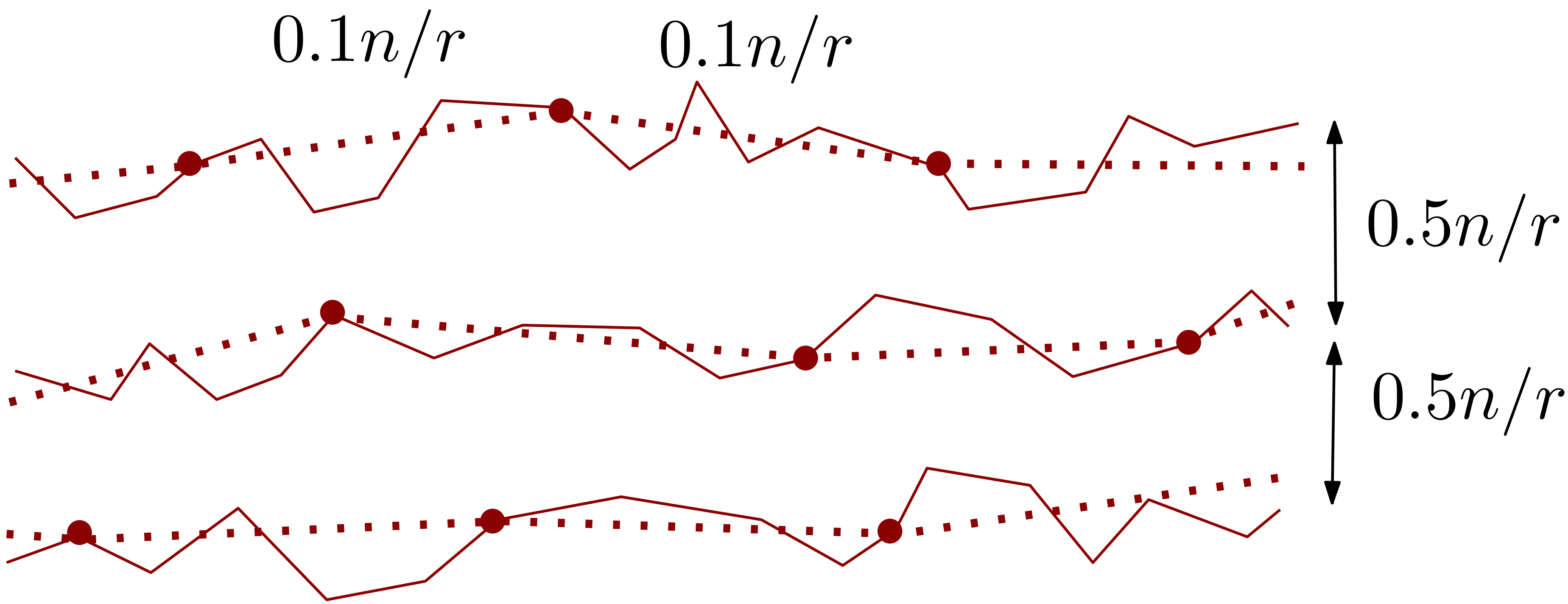
Def'n: level of pt q = # lines below q

Def'n: k-level = all pts at level k



Matoušek: "The Level Method"

1. For a fixed j, take all levels $\equiv j \pmod{0.5n/r}$
2. Simplify each such level s.t. each edge crosses exactly $0.1n/r$ lines



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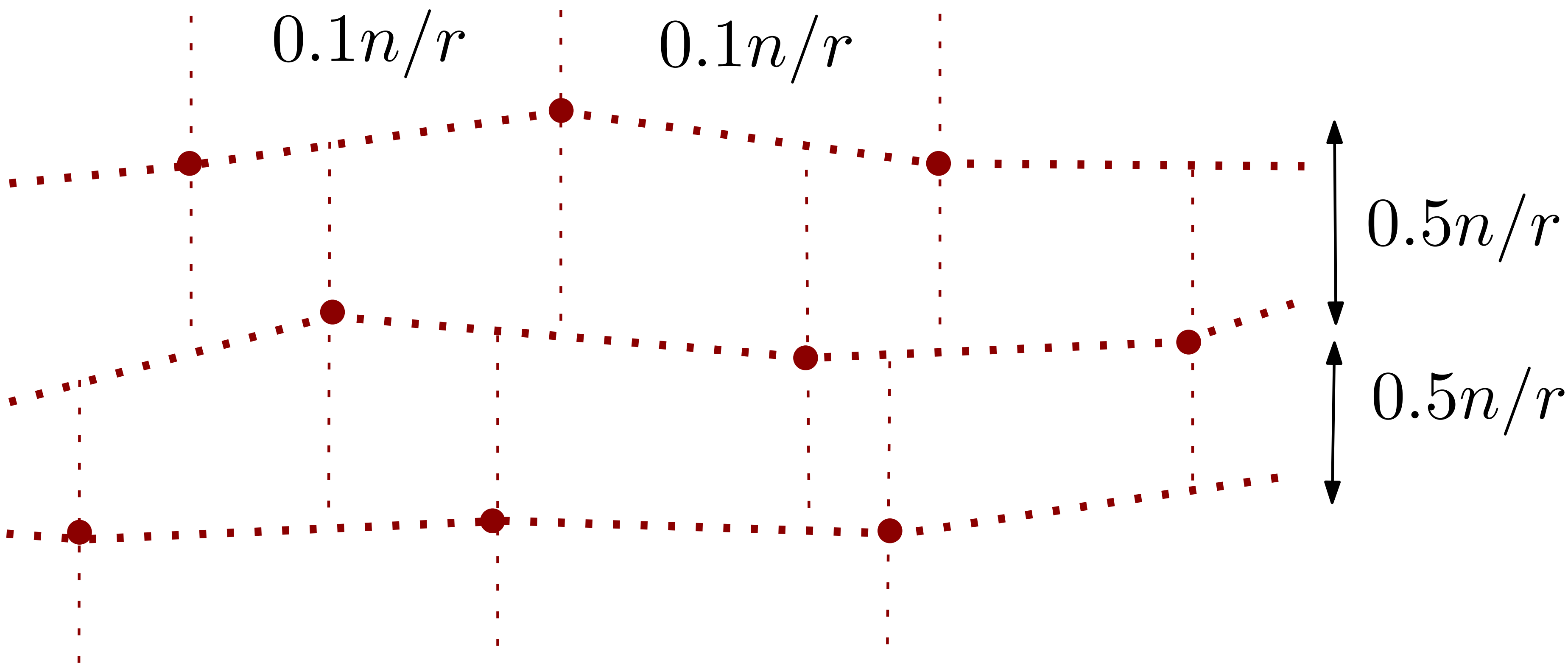
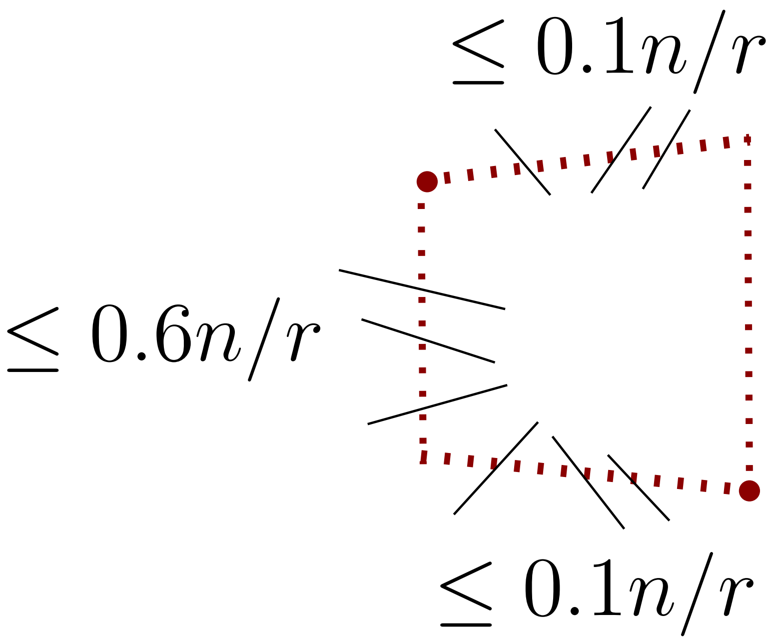


Matoušek: "The Level Method"

Analysis:

- each cell intersects $\leq 0.8n/r$ lines



Matoušek: "The Level Method"

Analysis:

- size $O\left(\frac{X^{(j)}}{0.1n/r}\right)$
  where $X^{(j)} =$ # vertices at levels $\equiv j \pmod{0.5n/r}$
  but don't know how big $X^{(j)}$ is... pick min j!

$$\Rightarrow\ O\left(\frac{1}{0.5n/r}\sum_j \frac{X^{(j)}}{0.1n/r}\right)$$

$$=\ O\left(\frac{1}{0.5n/r}\left(\frac{n^2}{0.1n/r}\right)\right)\ =\ \boxed{O(r^2)}$$

(simple & deterministic!)

Matoušek: "The Level Method"

Remarks:

- only works in 2D
- time? trivially polynomial
- Matoušek showed how to get time $\boxed{O(nr^2 \log r)}$ (but complicated!)

A Straightforward Recursion Method

Cut(L, n, r, Δ_0):

1. compute $(1/b)$-cutting K inside Δ_0 for large const b $\leftarrow$ by Matoušek's complicated method
2. for each cell Δ of K:

   Cut$(L_\Delta, n/b, r/b, \Delta)$

- size $S(r) \leq O(b^2)\, S(r/b) \quad \Rightarrow \boxed{O(r^{2+\varepsilon})}$
- time $T(n, r) \leq O(b^2)\, T(n/b, r/b) + O(nb^2 \log b)$ $\Rightarrow \boxed{O(nr^{1+\varepsilon})}$

  [r^ε factors improvable to polylog by nonconst b]

Agarwal, “A deterministic algorithm for partitioning arrangements of lines and its applications”, SoCG’89

Agarwal, *Intersection and Decomposition Algorithms for Planar Arrangements”*, PhD thesis, ’91

- $(1/r)$-cutting of size $O(r^2)$ in time $\boxed{O(nr \log n \log^{3.33} r)}$

  [based on the level method, also complicated. . .]

Matoušek, “Cutting hyperplane arrangements”, SoCG’90

Matoušek, “Approximations and optimal geometric divide-and-conquer”, STOC’91

- $(1/r)$-cutting of size $O(r^2)$ in time $\boxed{O(nr)}$ (finally!)

  [complicated
  suboptimal in higher D for large r
  requires notion of “ε-approximations”. . .]

We say that a collection A of hyperplanes is an *ε-approximation* for H provided that, for every segment e, it is

$$\left| \frac{|A_e|}{|A|} - \frac{|H_e|}{|H|} \right| < \varepsilon,$$

where A_e (resp. H_e) denotes the set of all hyperplanes of A (resp. of H) intersecting the segment e.

Chazelle, "An optimal convex hull algorithm and new results on cuttings", FOCS'91

- $(1/r)$-cutting of size $O(r^2)$ in time $\boxed{O(nr)}$ (again)

  [also optimal in higher D
  "hierarchical", useful in some appl'ns
  requires "ε-approximations", "sparse ε-nets", $\ldots$]

The next definition is adapted from [28]. We say that a subset R of H is (1/r)-*approximation* for H if, for any line segment e, the densities in R and H the hyperplanes crossing e differ by less than $1/r$, or, formally,

$$\left| \frac{|H_{|e}|}{|H|} - \frac{|R_{|e}|}{|R|} \right| < \frac{1}{r}.$$

ques of [15] and [29]. A subset R of H is called a (1/r)-*net* for H if, for any line segment e, the inequality $|H_{|e}| > n/r$ implies that $|R_{|e}| > 0$. A $(1/r)$-net plays the

We need to strengthen the notion of a $(1/r)$-net a little by requiring that the facial complexity of the portion of the arrangement that it forms within a given d-dimensional simplex s is not too large. We say that a $(1/r)$-net R is *sparse* for s if

$$\frac{v(R;s)}{v(H;s)} \le 4\left(\frac{|R|}{|H|}\right)^d.$$

Lemma 2.1 (Vertex-Count Estimation). *Let R be a $(1/r)$-approximation for a finite set H of hyperplanes in E^d. For any d-dimensional simplex s, we have*

$$\left|\frac{v(H;s)}{|H|^d} - \frac{v(R;s)}{|R|^d}\right| < \frac{1}{r}.$$

C. & Tsakalidis, not yet published, '14

- $(1/r)$-cutting of size $O(r^2)$ in time $\boxed{O(nr)}$ (yet again)

  ["re-interpretation" of Chazelle
  easier to understand (hopefully). . .]

Rest of Talk

I. History

II. “New” Alg’m

III. Coda

Prerequisite

Only Fact Needed: $(1/b)$-cutting of const size (don't care!)
in linear time for const b

- known already by Megiddo/Dyer!

Cuttings will be trapezoidal decompositions of line segments...

A New Recursion Method

Cut(L, n, r, Δ_0):

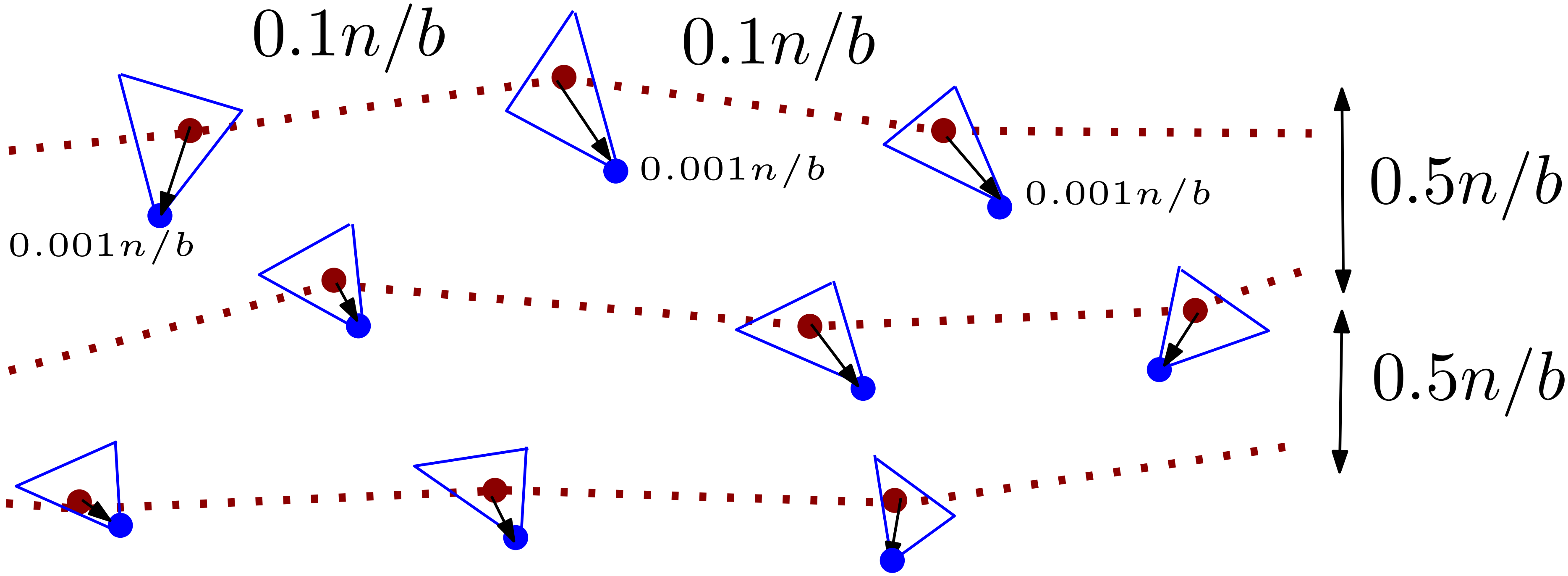
1. compute $(1/1000b)$-cutting G of const size (don't care!)
   for large const b $\leftarrow$ by Megiddo/Dyer

2. compute the best $(1/b)$-cutting K inside Δ_0 that is aligned to G, i.e., formed by line segments w. endpts from G $\leftarrow$ by BRUTE FORCE!

3. for each cell Δ of K:
   Cut($L_\Delta, n/b, r/b, \Delta$) (conceptually simple!)

Analysis of New Method

- consider the $(1/b)$-cutting K^* produced by the level method
- align it by "rounding" to vertices of G



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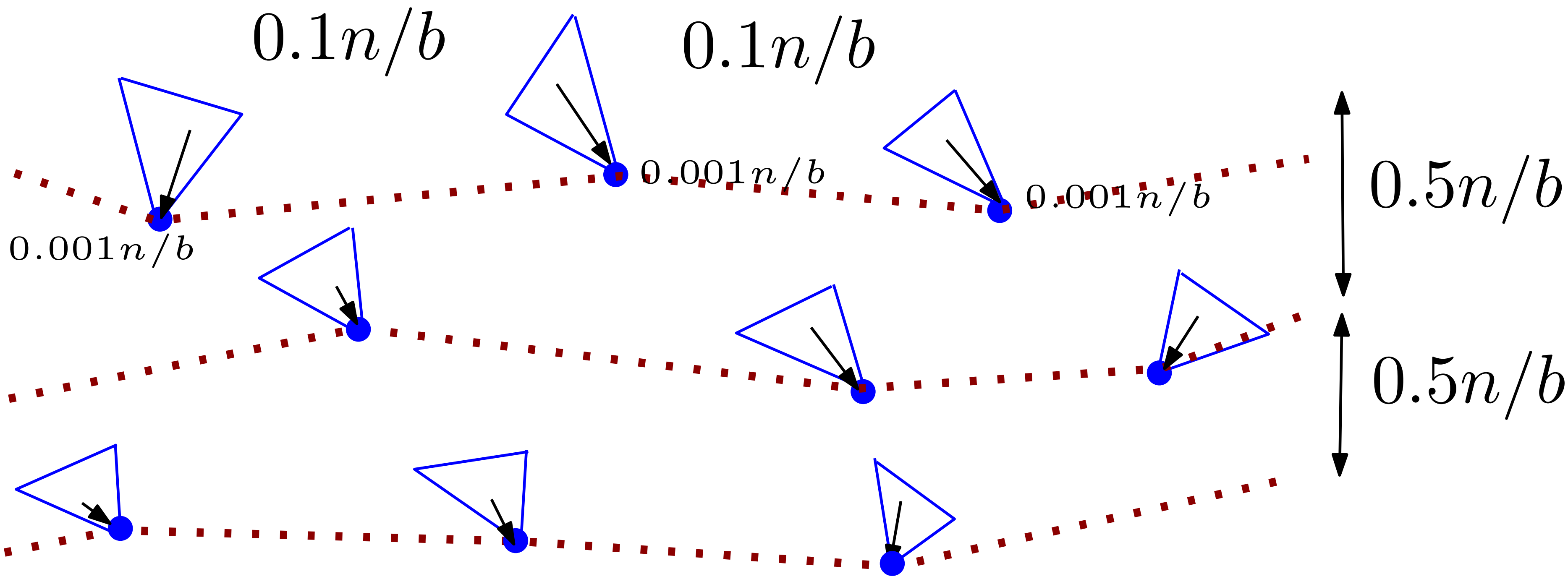

Recall Analysis of the Level Method. . .

- size $O\left(\frac{X^{(j)}}{0.1n/b}+Y^{(j)}\right)$
  where $X^{(j)} =$ # vertices at levels $\equiv j \pmod{0.5n/b}$
  inside Δ_0
  and $Y^{(j)} =$ # vertices at levels $\equiv j \pmod{0.5n/b}$
  on boundary of Δ_0
  pick min j!

$$\Rightarrow O\left(\frac{1}{0.5n/b}\sum_j\left(\frac{X^{(j)}}{0.1n/b}+Y^{(j)}\right)\right)$$

$$= O\left(\frac{1}{0.5n/b}\left(\frac{X}{0.1n/b}+n\right)\right) = \boxed{O(\tfrac{X}{n^2}b^2+b)}$$

where $X =$ total # intersections inside Δ_0

Back to Analysis of New Method

$\Rightarrow$ size of K^* (from the level method) $= O(\frac{X}{n^2}b^2 + b)$

$\Rightarrow$ size of K (from our brute force) $= O(\frac{X}{n^2}b^2 + b)$

$\Rightarrow$ overall size

$$\boxed{S(n, r, X) = \sum_{i=1}^{O((X/n^2)b^2+b)} S(n/b, r/b, X_i)}$$

where $\sum_i X_i = X$

Solving the Recurrence

$$S(n, r, X) = \sum_{i=1}^{O((X/n^2)b^2+b)} S(n/b, r/b, X_i)$$

- guess... $S(n, r, X) \leq \frac{X}{n^2} f(r) + g(r)$

$$\Rightarrow \text{ RHS } \leq \sum_{i=1}^{O((X/n^2)b^2+b)} \left(\frac{X_i}{(n/b)^2} f(r/b) + g(r/b) \right)$$
$$\leq \frac{X}{n^2} b^2 f(r/b) + O\left(\frac{X}{n^2} b^2 + b\right) g(r/b)$$

- set $f(r) = b^2 f(r/b) + O(b^2 g(r/b))$
$g(r) = O(b)\, g(r/b)$ $\Rightarrow O(r^{1+\varepsilon})$

Solving the Recurrence

$$S(n, r, X) = \sum_{i=1}^{O((X/n^2)b^2+b)} S(n/b, r/b, X_i)$$

- guess... $S(n, r, X) \leq \frac{X}{n^2} f(r) + g(r)$

$$\Rightarrow \text{ RHS } \leq \sum_{i=1}^{O((X/n^2)b^2+b)} \left(\frac{X_i}{(n/b)^2} f(r/b) + g(r/b) \right)$$

$$\leq \frac{X}{n^2} b^2 f(r/b) + O\left(\frac{X}{n^2} b^2 + b\right) g(r/b)$$

- set $f(r) = b^2 f(r/b) + O(r^{1+\varepsilon}) \quad \Rightarrow \quad \boxed{O(r^2)}$

Analysis of New Method (Cont'd)

- overall time

$$\boxed{T(n, r, X) = \sum_{i=1}^{O((X/n^2)b^2+b)} T(n/b, r/b, X_i) + O(n)}$$

$\Rightarrow$ $\boxed{O(nr)}$ similarly

Rest of Talk

I. History

II. “New” Alg’m

III. Coda

Remarks on "New" Method

- similar to Chazelle

  [hierarchical, but no prerequisites on "ε-approximations" or "ε-nets"]

- similar flavor as approx alg'ms/PTASes

  [comparing with OPT, rounding OPT, using brute force for small subproblems, . . .]

- HUGE consts!

  [see Har-Peled'98 for more practical, rand. implementations]

- does not generalize to higher D

- leads to new results on shallow cuttings. . .

Shallow Cuttings

Def'n: A k-shallow $(1/r)$-cutting is a subdivision covering all pts of level $\leq k$ s.t. each cell intersects $\leq n/r$ lines

Theorem: In 2D, $\exists\ \Omega(n/r)$-shallow $(1/r)$-cutting of size $O(r)$

- existence proof by the sampling method (Matoušek'91) or the level method
- Ramos'99: $O(n \log r)$ randomized time
- C.&Tsakalidis'14: $O(n \log r)$ deterministic time [similar ideas, can compare with general optimal K^*]

Shallow Cuttings

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Theorem: In 3D, $\exists\ \Omega(n/r)$-shallow $(1/r)$-cutting of size $O(r)$

- existence proof by the sampling method (Matoušek'91)
- Ramos'99: $O(n \log r)$ randomized time
- C.&Tsakalidis'14: $O(n \log r)$ deterministic time [more complicated. . .]

Open Problems

1. Hopcroft's problem in $O(n^{4/3})$ time w/o iterated log?
2. PTAS for min-size $(1/r)$-cutting (in $O(nr)$ time)?
3. deterministic shallow cutting with $O(r^{\lfloor d/2 \rfloor})$ size for odd $d \geq 5$?
4. faster deterministic cutting if don't need conflict lists?
   [known: $O(n \log r)$ for $r \leq n^{\alpha}$]
5. const factors
   [Matoušek'98: size $\approx 4r^2$]
6. best const for $(1/2)$-cutting? [Har-Peled'98]
   $(3/4)$-cutting of size 4, $(2/3)$-cutting of size 6 [??]
   $\Rightarrow$ $(1/2)$-cutting of size 24