Conflict-Free Coloring of Points w.r.t. Rectangles

& Approximation Algorithms for Discrete Independent Set

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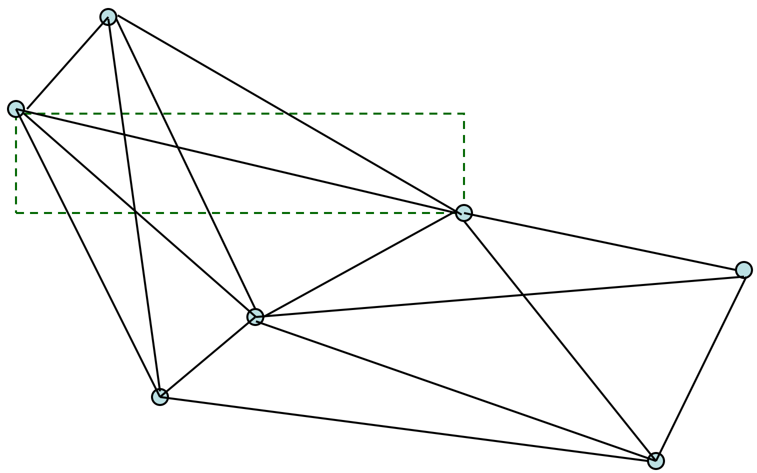
A Problem in Combinatorial Geometry

- Given n pts in 2D, define rectangle Delaunay graph:
  pq is an edge iff the smallest axis-aligned rectangle enclosing p,q is empty of pts



- Does there exist a large independent set in this graph?

[for standard Delaunay triangulation: $\Omega(n)$...]

$\exists$ Indep. Set of Size $\Omega(n^{1/2})$

Simple Proof 1:

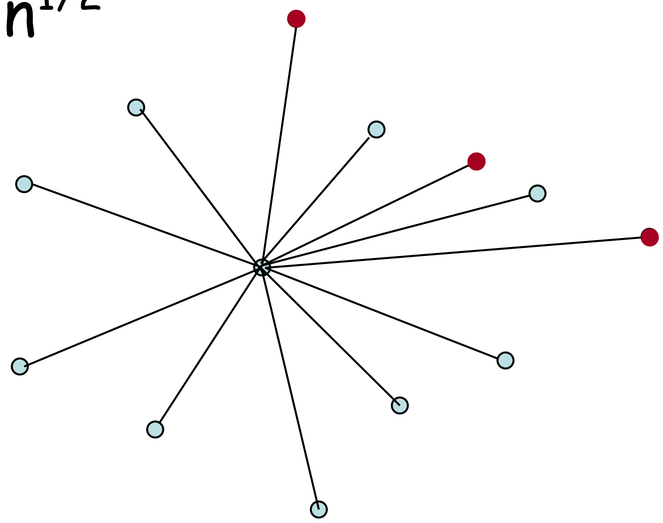
- Case 1: max degree $\leq n^{1/2}$

  Run greedy alg'm $\Rightarrow$ done

- Case 2: $\exists$ vertex v with degree $> n^{1/2}$

  $\Rightarrow$ $\exists$ monotone chain of size $\Omega(n^{1/2})$

  $\Rightarrow$ done



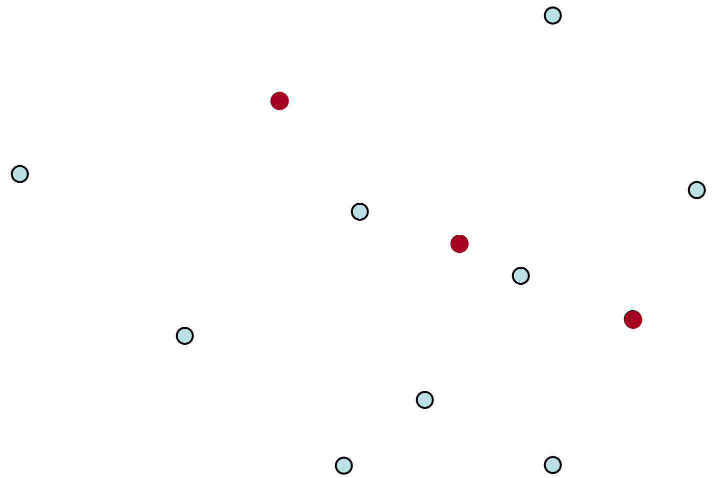
$\exists$ Indep. Set of Size $\Omega(n^{1/2})$

Simple Proof 2:

- By Erdős-Szekeres Thm,

  $\exists$ monotone increasing or decreasing chain of size $\Omega(n^{1/2})$

  $\Rightarrow$ done



History

- Har-Peled, Smorodinsky'03: $\Omega(n^{1/2})$
- Pach, Tardos/Alon/...'03: $\Omega(n^{1/2} \log^{1/2} n)$
- Ajwani, Elbassioni, Govindarajan, Ray'07: $\Omega(n^{0.618})$

  [exponent ≈ golden ratio - 1]

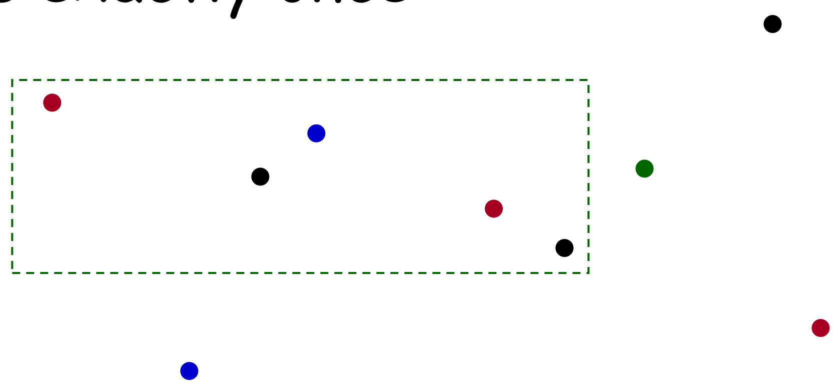
- New Result: $\Omega(n^{0.632})$

  [exponent obtained using computer calculations...]

- Chen, Pach, Szegedy, Tardos'09: $O(n \log^2 \log n / \log n)$

Motivation 1: Conflict-Free Colorings

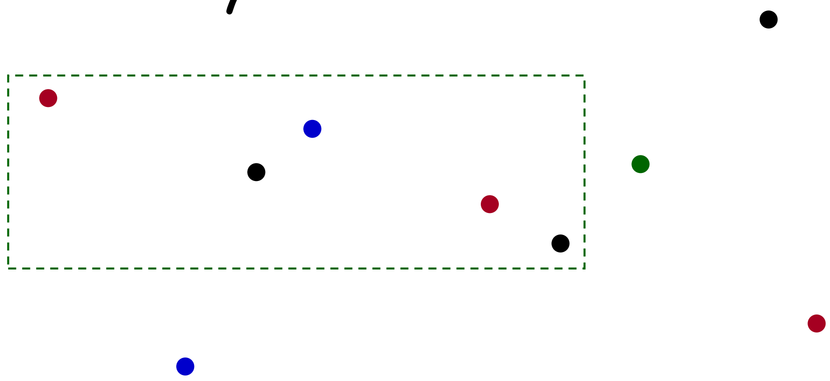
- Given n pts in 2D, color them s.t.
  every axis-aligned rectangle contains a color that occurs exactly once



- reduces to indep. set in Delaunay graph
  [Even,Lotker,Ron,Smorodinsky'02:
  find indep. set, make it a new color class, remove, & repeat]
- # colors = O^*(n / max indep. set size)

Motivation 1: Conflict-Free Colorings

- Given n pts in 2D, color them s.t. every axis-aligned rectangle contains a color that occurs exactly once



- Ajwani,Elbassioni,Govindarajan,Ray'07: $O(n^{0.382})$
- New Result: $O(n^{0.368})$

Motivation 2: Approx. Alg'ms for Discrete Independent Set

- Given n pts & m axis-aligned rectangles in 2D,
  choose max subset of pts s.t.
  each given rectangle contains ≤ 1 chosen pt

  [similar to problems from Ene,Har-Peled,Raichel's Sunday talk]

- New Result: $O(n^{0.368})$-factor approx. alg'm

  [roughly: solve standard LP relaxation,
  draw random sample R with LP values as weights,
  & find indep. set in a Delaunay-like graph of R]

Motivation 2: Approx. Alg'ms for Discrete Independent Set

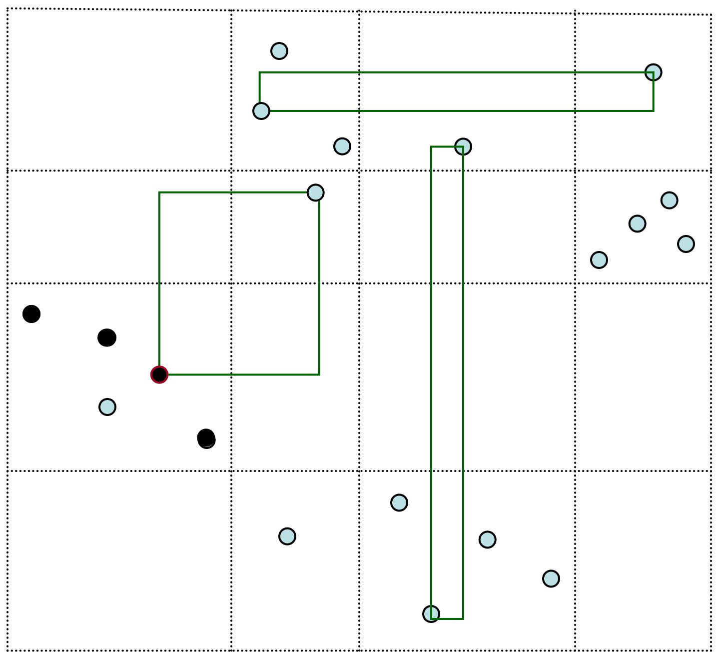
- OR: Given n axis-aligned boxes & m pts in R^d,
  choose max subset of boxes s.t.
  each given pt is contained in ≤ 1 chosen box

- Known Approx. Factor:
  - continuous version: O(loglog n) in 2D
    O(polylog n) for $d \geq 3$
  - discrete version: O(polylog n) in 2D,3D [Ene et al.]
    nothing for $d \geq 4$

- New Result: $O(n^{1 - 0.632/[2^{2d-3} - 0.368]})$ for $d \geq 4$

Previous Proof Sketch [Ajwani,Elbassioni, Govindarajan,Ray'07]

- use r x r grid, where each column/row contains n/r pts

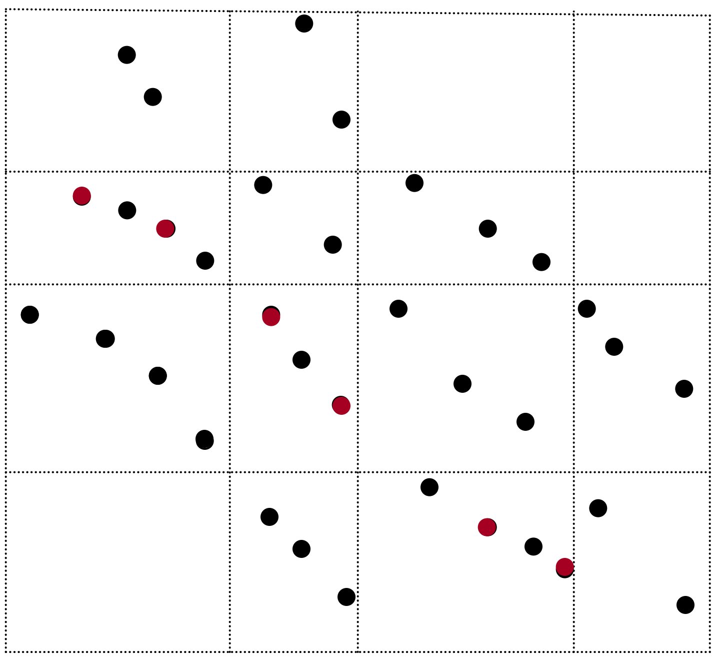


3 types of rectangles:

1. contains a grid pt
2. inside a column
3. inside a row

Previous Proof Sketch (Cont'd)

1. let M = set of all maxima inside all grid cells
   if $|M| \geq n/2$ then



$O(r)$ monotone chains

$\Rightarrow$ $\exists$ chain with $\Omega(n/r)$ pts

$\Rightarrow$ done

else discard M $\Rightarrow$ # pts = $\Omega(n)$

Previous Proof Sketch (Cont'd)

2. inside each column, recursively find indep. set

$\Rightarrow$ # pts = $\Omega(rT(n/r))$

$\Rightarrow$ expects # pts per row = $\Omega^*([rT(n/r)] / r) = \Omega^*(T(n/r))$

3. inside each row, recursively find indep. set of all pts found in Step 2

$\Rightarrow$ final # pts = $\Omega^*(rT(T(n/r)))$

[Technicality: need to randomize things!
generalize problem to coloring, & randomly pick a color class]

Previous Proof Sketch (Cont'd)

- Recurrence:

$$T(n) \geq \Omega^*(\min\{\ n/r,\ \ rT(T(n/r))\ \})$$

- Assume $T(n) \geq \Omega^*(n^a)$

  $\Rightarrow\ T(n) \geq \Omega^*(\min\{ n/r,\ r\,(n/r)^{a^2} \})$

  $\Rightarrow\ T(n) \geq \Omega^*(n^{a'})$ where $a' = 1 - (1 - a^2)/(2 - a^2)$

  $\Rightarrow\ a \approx 0.618$

Idea for Improvement

- Note: if p,q lie in same column & same row, the pair is "handled" twice (both Steps 2 and 3)!
- Fix: in Step 2's recursion inside a column, may ignore pairs p,q that lie inside common row
- Assume input already divided into h rows
- New Recurrence:

$$T(n, h) \geq \Omega^*(\min\{ n/r, \; rT(T(n/r, r), h/r) \})$$

Idea for Improvement (Cont'd)

- $T(n, h) \geq \Omega^*(\min\{ n/r, \; rT(T(n/r, r), h/r) \})$ (1)
  $T(n, h) \geq \Omega^*(\min\{ n/r, \; rT(T(n/r, h/r), r) \})$ (2)
  ["Base case": $T(n,h) \geq n/h$]

- Ex: $T(n,n) \geq \Omega^*(\min\{n^{0.622}, n^{0.378} [T(n^{0.622},n^{0.378})]^{0.618})$ by (1)
  $T(n^{0.622},n^{0.378}) \geq \Omega^*(\min\{n^{0.395}, n^{0.227} [T(n^{0.395},n^{0.151})]^{0.618}\})$ by (2)
  $T(n^{0.395},n^{0.151}) \geq \Omega^*(\min\{n^{0.272}, n^{0.123} [T(n^{0.272},n^{0.028})]^{0.618} \})$ by (2)
  $T(n^{0.272},n^{0.028}) \geq \Omega^*(n^{0.244})$ by base case
  $\Rightarrow T(n,n) \geq \Omega^*(n^{0.622})$ [improvement over $n^{0.618}$!]
  repeat $\Rightarrow T(n,n) \geq \Omega^*(n^{0.624})$

Further Improvement

- Add one more parameter: assume input already divided into h rows and v columns
- Use recursion in Step 1 too
- New New Recurrence:

$$T(n,v,h) \geq \Omega^{*}(\min_{r' \geq r}\{n/r,\ r'\,T(T(n/r',v/r',r'),r',h/r')\})$$

How to Solve this Recurrence?

- $T(n,v,h) \geq \Omega^*(\min_{r' \geq r}\{ n/r,\ \ r' T(T(n/r',v/r',r'),r',h/r') \})$
  ["Base cases": $T(n,v,h) \geq n/v$,
  $T(n,v,h) = T(n,h,v)$]

- Assume $T(n,v,h) \geq \Omega^*(n^{a_1} v^{b_1} h^{c_1})$
  $T(n,v,h) \geq \Omega^*(n^{a_2} v^{b_2} h^{c_2})$

$\Rightarrow T(n,v,h) \geq \Omega^*(n^a v^b h^c)$
where $(a,b,c) = (1+(a_1a_2-1)/d,\ b_1a_2/d,\ c_2/d)$
with $d = 2 - (a_1+b_1-c_1)a_2 + b_2 - c_2$ if $d \geq 1$

… A Math Problem

- So, define

  $g((a_1,b_1,c_1),(a_2,b_2,c_2)) := (1+(a_1a_2-1)/d,\ b_1a_2/d,\ c_2/d)$

  with $d = 2 - (a_1+b_1-c_1)a_2 + b_2 - c_2$ if $d \geq 1$

  $s(a,b,c) := (a,c,b)$

- Question: Starting with $v = (0,0,0)$, $u = (1,-1,0)$ in R^3, apply operators *g* & *s* repeatedly in any order. Get $(a,0,0)$ with largest *a*?

- Ex: Repeat $v := g(\ s(\ g(g(u,v),v)\ ),\ v)$ $\Rightarrow$ $(0.624,0,0)$

Proof of a ≥ 0.631

```
w0 := s(u); w1 := g(u,w0); w2 := g(u,w1); w3 := g(u,w2);
w4 := g(u,w3); z0 := s(w1); z1 := g(u,z0); z2 := g(u,z1);

v :=
g(s(g(g(g(g(u,g(u,g(w3,g(w2,s(z1))))),g(g(u,g(w3,g(w2,s(z1
)))),s(g(g(z2,s(g(w2,z0))),s(g(w3,g(w2,s(z1)))))))),s(g(g(
g(z3,g(z2,s(g(u,s(z1))))),g(g(z2,s(g(z2,s(z1)))),s(g(w3,g(
z2,s(z1)))))),s(g(g(w4,g(w3,g(z2,s(z1)))),s(g(g(z2,s(g(w2,
s(z1)))),s(g(w3,g(z2,s(z1))))))))))),v)),v);
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Proof of $a \geq 0.632$

```
w0 := s(u); w1 := g(u,w0); w2 := g(u,w1); w3 := g(u,w2); w4 := g(u,w3); w5 := g(u,w4);
z0 := s(w1); z1 := g(u,z0); z2 := g(u,z1); z3 := g(u,z2); z4 := g(u,z3); z5 := g(u,z4);
v := \
g(s(g(g(g(g(g(z5,g(z5,g(z4,g(g(u,g(u,g(u,s(g(w1,w0))))),s(g(g(w2,s(z1)),s(z2))))))),g(g(z5,g(z4\
,g(g(u,g(u,g(w3,s(g(w1,w0))))),g(g(u,g(w3,s(g(w1,s(w2))))),s(g(g(w2,g(w1,s(w2))),s(g(u,s(g(w1,w\
0)))))))))),g(g(g(u,g(u,g(u,g(w3,s(g(w1,w0)))))),g(g(u,g(u,g(w3,s(g(w1,s(w2)))))),g(g(w3,g(w3,s\
(g(z1,s(w2))))),s(g(g(w2,g(z1,s(w2))),s(g(u,s(g(w1,s(w2))))))))))),g(g(g(u,g(w3,g(w3,s(g(z1,s(w2\
)))))),g(g(w4,g(z3,g(z2,s(g(w2,s(z1)))))),s(g(g(w3,g(w2,s(z1))),s(g(u,((s(g(w1,w0)))))))))),s(g\
(g(g(w3,g(w2,g(z1,s(w2)))),g(g(z2,g(z1,s(w2))),s(g(w3,s(g(z1,s(w2))))))),s(g(g(u,g(w3,s(g(z1,s(\
w2))))),s(g(g(w2,g(z1,s(w2))),s(g(u,s(g(w1,s(w2)))))))))))))),g(g(g(z5,g(g(u,g(u,g(u,g(w3,s(g(\
w1,w0)))))),g(g(u,g(u,g(w3,s(g(w1,s(w2)))))),g(g(w4,g(w3,g(z2,s(z1)))),s(g(g(w2,g(z1,s(w2))),s(\
g(u,s(g(w1,s(w2)))))))))),g(g(g(u,g(w4,g(w4,g(w3,s(g(w1,s(w2))))))),g(g(w4,g(w4,g(w3,g(w2,s(z1\
))))),g(g(w4,g(z3,g(z2,s(g(u,s(z1)))))),s(g(g(z2,s(g(w2,s(z1)))),s(g(w3,g(z2,s(z1))))))))),g(g(\
g(w4,g(w4,g(w3,g(z2,s(z1))))),g(g(w4,g(z3,g(z2,s(g(w2,s(z1)))))),s(g(g(z2,s(g(z2,s(z1)))),s(g(z\
3,g(z2,s(g(u,s(z1)))))))))),s(g(g(g(z3,g(z2,s(g(w2,s(z1))))),g(g(z2,s(g(z2,s(z1)))),s(g(w3,g(z2\
,s(z1)))))),s(g(g(w4,g(z3,g(z2,s(g(u,s(z1)))))),s(g(g(z2,s(g(w2,s(z1)))),s(g(w3,g(z2,s(z1))))))\
))))))),s(g(g(g(g(g(u,g(u,g(w2,z0))),g(g(w3,g(w2,s(z1))),g(g(w2,s(g(w2,z0))),s(g(w3,g(w2,s(z1))\
))))),g(g(g(w3,g(w2,s(g(w2,z0)))),g(g(z2,g(w2,s(g(u,((w1)))))),s(g(u,((s(g(w1,w0)))))))),s(g(g(\
u,g(u,((g(w2,z0))))),s(g(g(w2,s(g(w2,z0))),s(g(w3,g(w2,s(z1)))))))))),s(g(g(g(u,g(u,g(w3,g(w2,s\
(z1))))),g(g(w4,g(w3,g(z2,s(z1)))),s(g(g(z2,s(g(w2,z0))),s(g(w3,g(w2,s(z1))))))))),s(g(g(g(w3,g(\
w2,s(z1))),g(g(w2,s(g(w2,z0))),s(g(w3,g(w2,s(z1)))))),s(g(g(u,g(w3,g(w2,s(z1)))),s(g(g(w2,s(z1)\
),s(g(u,g(w2,z0))))))))))),s(g(g(g(u,g(u,g(u,g(u,s(g(w1,w0)))))),g(g(u,g(u,g(u,s(g(w1,s(w2))))\
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g(z2,s(z1)))),s(g(g(w2,s(g(w2,z0))),s(g(w3,g(w2,s(z1)))))))))))))))),s(g(g(g(g(g(z3,g(g(u,g(u,\
g(w2,z0))),g(g(w3,g(w2,s(z1))),s(g(z2,s(g(w2,z0))))))),g(g(g(w3,g(w3,g(w2,s(z1)))),g(g(w3,g(w2,\
g(z1,s(w2)))),s(g(g(u,g(w2,z0)),s(g(w2,s(z1))))))),g(g(g(w3,g(w2,g(z1,s(w2)))),g(g(z2,g(z1,s(w2\
))),s(g(w3,s(g(z1,s(w2))))))),s(g(w3,((s(g(g(w2,z0),s(z2)))))))))),g(g(g(g(w3,g(w3,g(w2,s(z1)))\
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,s(z1))))),g(g(z2,g(z1,s(w2))),s(g(u,((((z1)))))))),s(g(g(u,g(w3,((g(w2,s(z1)))))),s(g(g(z2,s(g\
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Proof of a ≥ 0.632 (Cont'd)

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(w2,s(z1)))),s(g(w3,g(z2,s(z1))))))))),s(g(g(g(w4,g(w4,g(w3,g(z2,s(z1))))),g(g(w4,g(w3,g(w2,s(\
z1)))),s(g(g(z2,s(g(w2,s(z1)))),s(g(w3,g(z2,s(z1))))))),s(g(g(g(w3,g(z2,s(g(w2,z0)))),g(g(z2,s\
(g(w2,s(z1)))),s(g(w3,g(z2,s(z1)))))),s(g(g(u,g(w3,g(w2,s(z1)))),s(g(g(z2,s(g(w2,z0))),s(g(w3,g\
(w2,s(z1))))))))))))),g(g(g(g(g(w3,g(w3,g(z2,s(z1)))),g(g(z3,g(z2,s(g(w2,s(z1))))),g(g(z2,s(g(\
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,s(g(w3,((s(g(z1,s(w2)))))))),s(g(g(w3,g(w3,((g(w2,s(z1)))))),s(g(g(z2,s(g(z2,s(z1)))),s(g(w3,\
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g(z2,s(g(z2,s(z1)))),s(g(w3,g(z2,s(z1))))))),s(g(g(g(z3,g(z2,s(g(w2,s(z1))))),g(g(z2,s(g(w2,s(\
z1)))),s(g(w3,g(z2,s(z1)))))),s(g(g(w4,g(w3,g(z2,s(z1)))),s(g(g(z2,s(g(w2,s(z1)))),s(g(w3,g(w2,\
s(z1))))))))))))),s(g(g(g(u,g(w4,g(w4,g(w3,s(g(z1,s(w2))))))),g(g(w4,g(w3,g(w3,s(g(z1,s(w2)))))\
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s(z1)))))))))),s(g(g(g(z3,g(z2,s(g(w2,s(z1))))),g(g(z2,g(w2,s(g(u,((w1)))))),s(g(w3,((s(g(z1,s(\
w2))))))))),s(g(g(w3,g(w3,((g(w2,s(z1)))))),s(g(g(z2,s(g(z2,s(z1)))),s(g(w3,g(z2,s(z1)))))))))\
))))),s(g(g(g(g(u,g(u,g(u,g(w3,g(w3,s(z1)))))),g(g(u,g(w4,g(w4,g(w3,s(g(w1,s(w2))))))),g(g(w4,g\
(w4,g(w3,s(g(z1,s(w2)))))),g(g(w4,g(w3,g(z2,s(z1)))),s(g(g(z2,g(z1,s(w2))),s(g(w3,s(g(z1,s(w2))\
))))))))),g(g(g(u,g(w4,g(w3,g(w3,s(g(g(u,((w0))),s(w2))))))),g(g(w4,g(w4,g(w3,g(w2,s(z1))))),g(\
g(w4,g(z3,g(z2,s(g(w2,s(z1)))))),s(g(g(z2,s(g(z2,s(z1)))),s(g(w3,g(z2,s(z1)))))))),g(g(g(z4,g(\
w4,g(w3,g(z2,s(g(u,s(z1))))))),g(g(w4,g(z3,g(z2,s(g(w2,s(z1)))))),s(g(g(g(u,g(u,s(z1))),s(g(z2,\
s(g(u,s(z1)))))),s(g(z3,g(z2,s(g(w2,s(z1))))))))))),s(g(g(g(z3,g(z2,s(g(z2,s(z1))))),g(g(z2,s(g(\
z2,s(z1)))),s(g(w3,g(z2,s(z1)))))),s(g(g(w4,g(w3,g(w2,s(z1)))),s(g(g(z2,s(g(z2,s(z1)))),s(g(w3,\
g(z2,s(z1))))))))))))),s(g(g(g(g(g(w3,g(w3,g(w2,s(z1)))),g(g(w3,g(z2,s(g(w2,z0)))),g(g(z2,s(g(w\
2,s(z1)))),s(g(w3,g(z2,s(z1)))))),g(g(g(w3,g(z2,s(g(w2,s(z1))))),g(g(z2,g(w2,s(g(u,((w1)))))),\
s(g(u,((s(g(w1,s(w2)))))))),s(g(g(w3,g(w3,((g(w2,s(z1)))))),s(g(g(z2,s(g(w2,s(z1)))),s(g(w3,g(\
z2,s(z1)))))))))),s(g(g(g(w4,g(w4,g(w3,g(z2,s(z1))))),g(g(w4,g(w3,g(w2,s(z1)))),s(g(g(z2,s(g(w2\
,s(z1)))),s(g(w3,g(z2,s(z1))))))),s(g(g(g(w3,g(z2,s(g(w2,z0)))),g(g(z2,s(g(w2,s(z1)))),s(g(w3,\
g(z2,s(z1)))))),s(g(g(w4,g(w3,g(z2,s(z1)))),s(g(g(w2,s(g(w2,z0))),s(g(w3,g(w2,s(z1))))))))))))\
,s(g(g(g(u,g(u,g(w3,g(w3,s(g(g(u,((u))),s(w2))))))),g(g(u,g(w3,g(w3,s(g(z1,s(w2)))))),g(g(w4,g(\
z3,g(z2,s(g(w2,s(z1)))))),s(g(g(w3,g(w2,s(z1))),s(g(u,((s(g(w1,w0)))))))))),g(g(g(w4,g(w4,g(w3\
,g(z2,s(z1))))),g(g(w4,g(z3,g(z2,s(g(w2,s(z1)))))),s(g(g(z2,s(g(z2,s(z1)))),s(g(w3,g(z2,s(z1)))\
))))),s(g(g(g(z3,g(z2,s(g(w2,s(z1))))),g(g(z2,s(g(z2,s(z1)))),s(g(w3,g(z2,s(z1)))))),s(g(g(w4,g\
(z3,g(z2,s(g(u,s(z1)))))),s(g(g(z2,s(g(w2,s(z1)))),s(g(w3,g(z2,s(z1))))))))))))))))),v)),v);
```

Open Problems

- Improve 0.632 ?
- Higher dimensions ?

  [current bound: $\Omega(n^{0.632/2^{d-2}})$]