Deterministic APSP, Orthogonal Vectors, and More:

Quickly Derandomizing Razborov–Smolensky

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Problem 1: All-Pairs Shortest Paths (APSP)

Given real-weighted graph with n vertices, compute the shortest path between every pair of vertices

- textbook: $O(n^3)$ time
- Big Open Q: $O(n^{3-\varepsilon})$??
- importance: many applications, & various APSP-hardness results...

History of APSP Alg'ms

Fredman'75	$n^3(\log\log n/\log n)^{1/3}$
Takaoka'92	$n^3\sqrt{\log\log n/\log n}$
Dobosiewicz'90	$n^3/\sqrt{\log n}$
Han'04	$n^3(\log\log n/\log n)^{5/7}$
Takaoka'04	$n^3\log^2\log n/\log n$
Zwick'04	$n^3\sqrt{\log\log n}/\log n$
Chan'05	$n^3/\log n$
Han'06	$n^3(\log\log n/\log n)^{5/4}$
Chan'07	$n^3(\log\log n)^3/\log^2 n$
Han–Takaoka'12	$n^3\log\log n/\log^2 n$
Williams [STOC'14]	$\boxed{n^3/2^{\Omega(\sqrt{\log n})}}$ rand. (Monte Carlo)
new	$\boxed{n^3/2^{\Omega(\sqrt{\log n})}}$ det.

Problem 2: Orthogonal Vectors (OV)

Given n Boolean vectors in d dims, is there a pair with dot product $= 0$?

- trivial: $O(dn^2)$ time
- improvable by rect. matrix multiplication (e.g., $O(n^{2+\varepsilon})$ for $d \ll n^{0.30}$, or $\widetilde{O}(n^2)$ for $d \ll n^{0.17}$)
- trivial: $O(2^d)$ time

- Big Open Q: $O(n^{2-\varepsilon})$ alg'm for some $d = \omega(\log n)$??

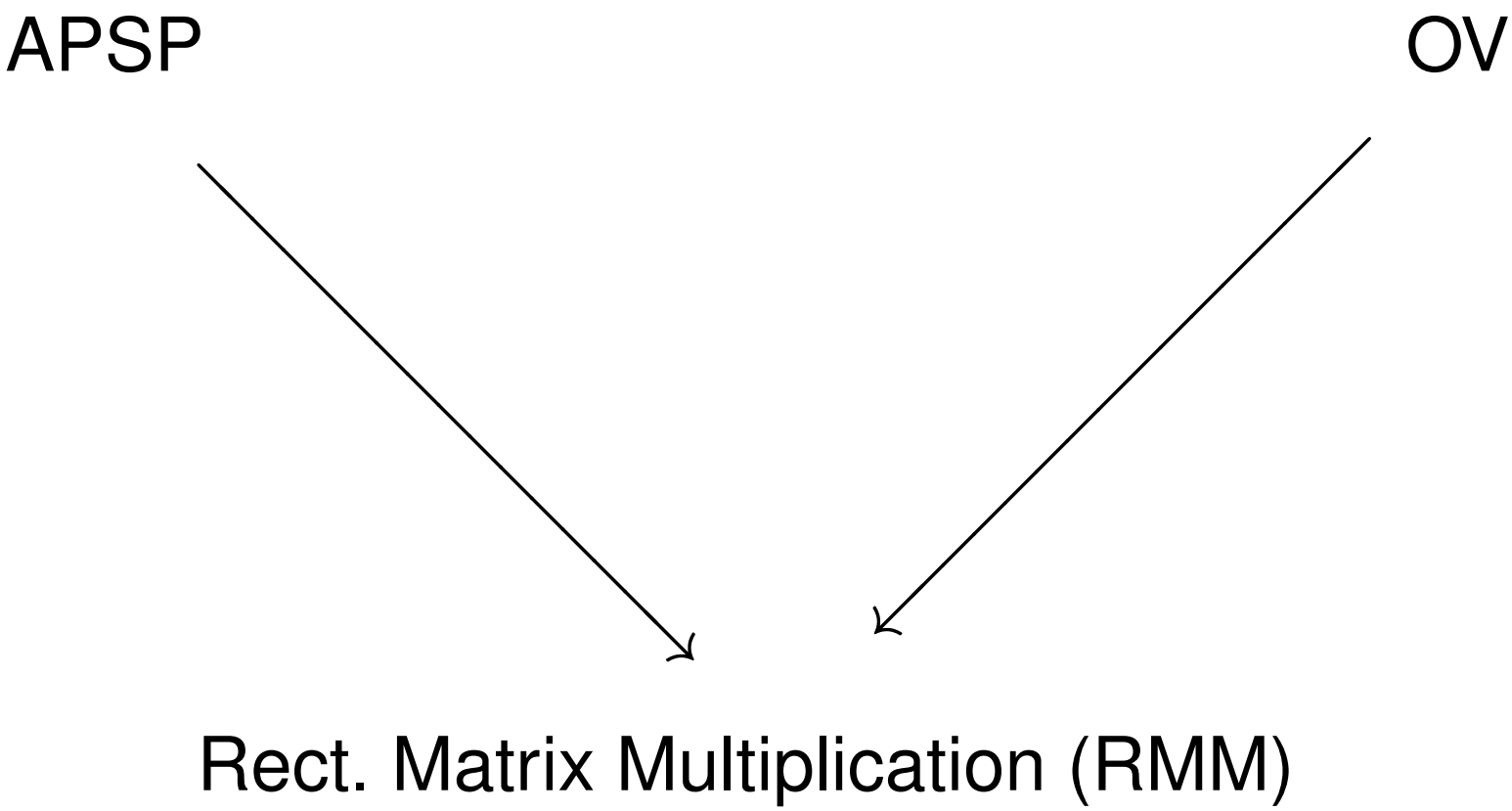
- importance: applications e.g. to k-SAT (OV is SETH-hard), & numerous OV-hardness results . . .

History of OV Alg'ms

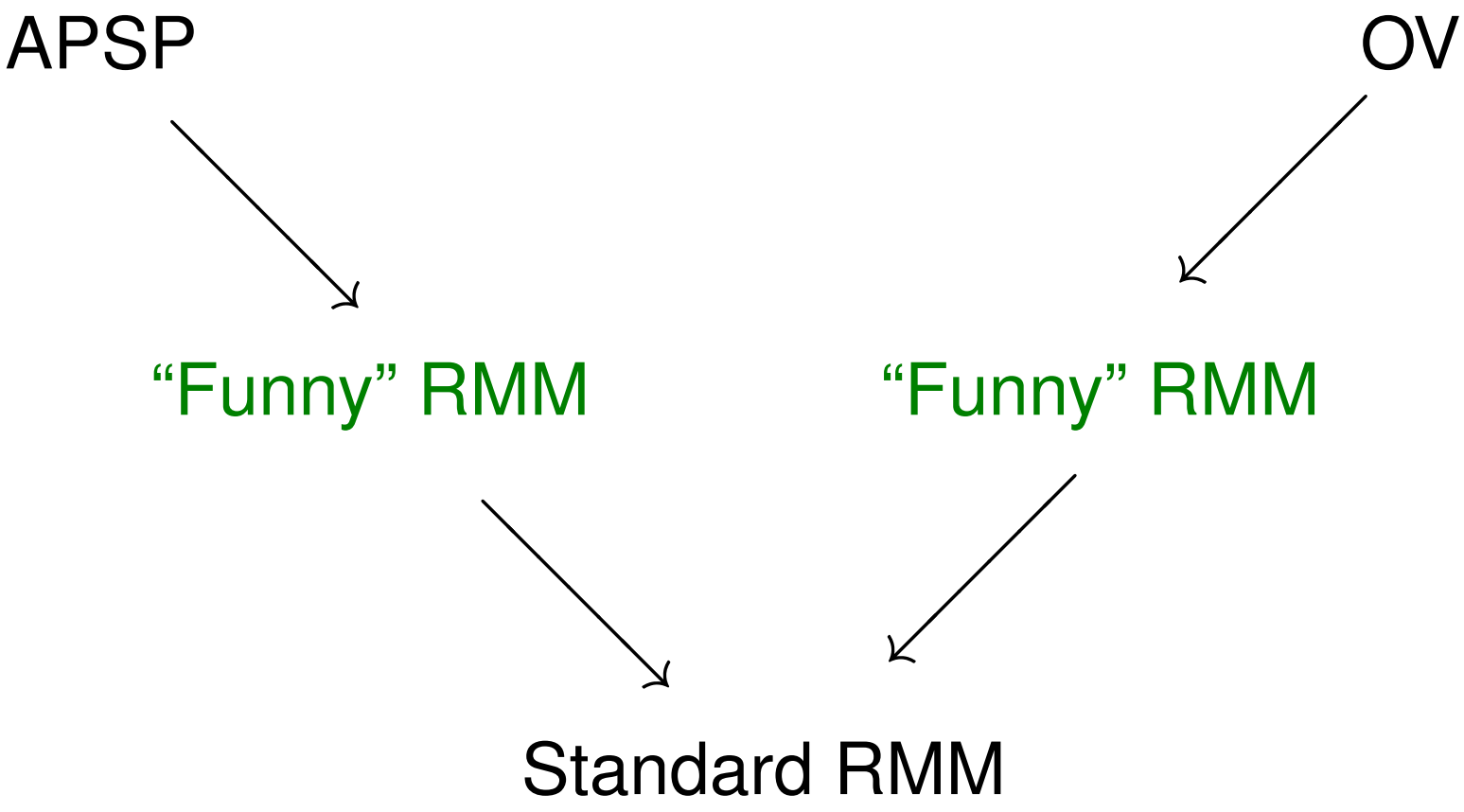
For $d = c \log n$:

Impagliazzo–Lovett–Paturi–Schneider'14	$n^{2-1/c^{O(1)}}$	
Abboud–Williams–Yu [SODA'15]	$\boxed{n^{2-1/O(\log c)}}$	rand. (Monte Carlo)
new	$\boxed{n^{2-1/O(\log c)}}$	det.

Plan



Plan



Standard RMM

Given vectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}$ in d dims, compute

$$\begin{pmatrix} — \mathbf{x}^{(1)} — \\ \\ \vdots \\ \\ — \mathbf{x}^{(n)} — \end{pmatrix} \begin{pmatrix} | & & | \\ \mathbf{y}^{(1)} & \cdots & \mathbf{y}^{(n)} \\ | & & | \end{pmatrix} = \begin{pmatrix} \\ \text{all dot products} \\ \mathbf{x}^{(s)} \cdot \mathbf{y}^{(t)} \\ \\ \end{pmatrix}$$

- $\widetilde{O}(n^2)$ time for $d \ll n^{0.17}$ [Coppersmith'82]

"Funny" RMM

Given vectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}$ in d dims, compute

$$\begin{pmatrix} — \, \mathbf{x}^{(1)} \, — \\ \\ \vdots \\ \\ — \, \mathbf{x}^{(n)} \, — \end{pmatrix} \begin{pmatrix} | & & | \\ \mathbf{y}^{(1)} & \cdots & \mathbf{y}^{(n)} \\ | & & | \end{pmatrix} = \begin{pmatrix} \\ \text{all values} \\ f(\mathbf{x}^{(s)}, \mathbf{y}^{(t)}) \\ \\ \end{pmatrix}$$

for some "funny" function $f(\mathbf{x}, \mathbf{y})$

- $\widetilde{O}(n^2)$ time for what d??

APSP...

- is known to be reducible to "funny" RMM with

$$f(\mathbf{x},\mathbf{y}) = \min_{k=1}^{d}(x_k + y_k) \qquad (\mathbf{x},\mathbf{y} \in \mathbb{R}^d)$$

  called min-plus RMM

- by known techniques (Fredman's subtraction trick, Matoušek's dominance method, ...), can be further reduced to a "less funny" RMM with

$$f(\mathbf{x},\mathbf{y}) = \bigwedge_{i=1}^{d'} \bigvee_{j=1}^{d''} (x_{ij} \wedge y_{ij}) \qquad (\mathbf{x},\mathbf{y} \in \{0,1\}^{d'd''})$$

  which I'll call AND-OR-AND RMM $(d', d'' = \text{poly}(d))$

  (Williams' original alg'm used XOR-AND-OR-AND RMM...)

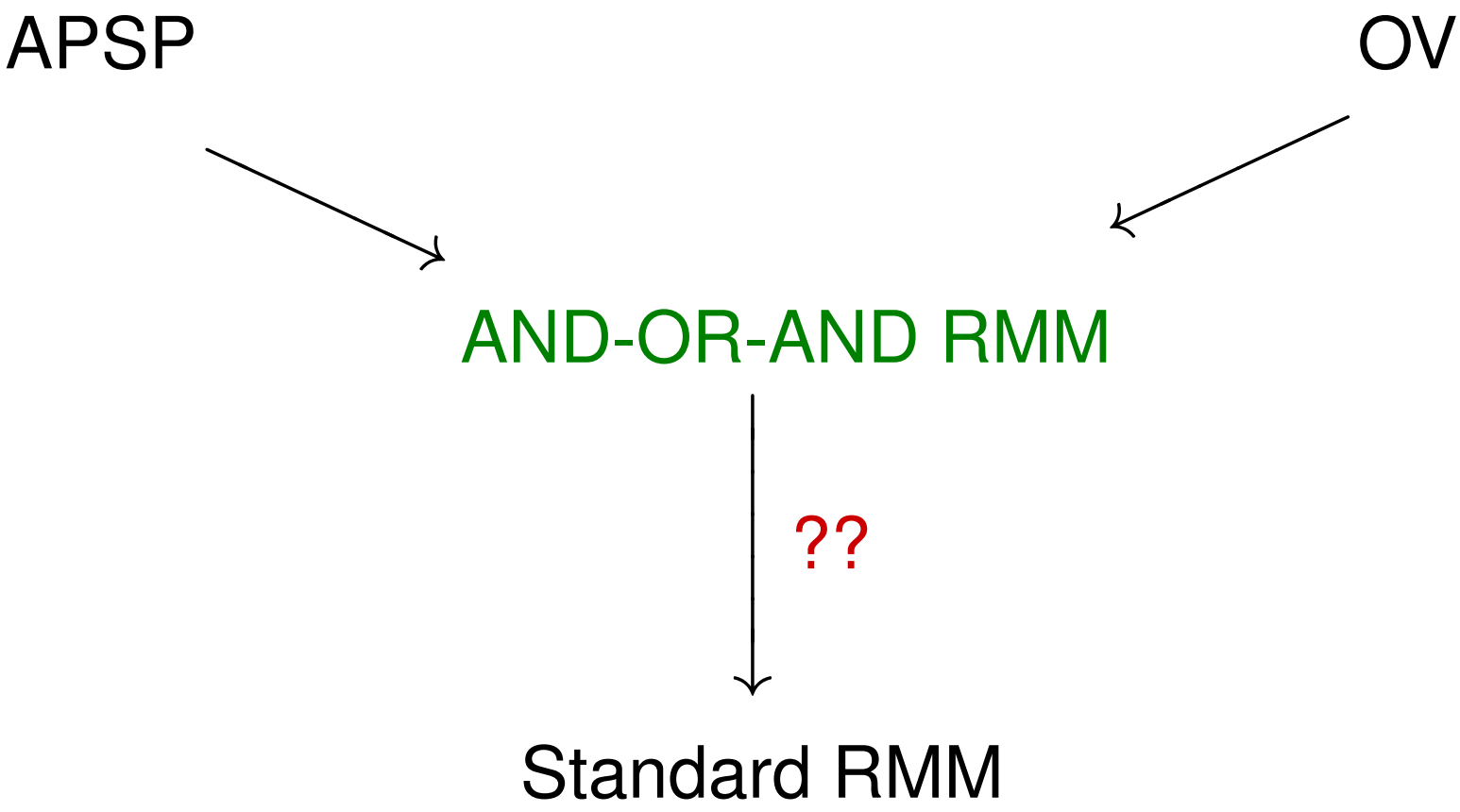
OV…

- by dividing into n/g groups of g vectors, can be reduced to "funny" RMM with

$$f(\mathbf{x},\mathbf{y}) = \bigwedge_{i=1}^{g} \bigwedge_{j=1}^{g} \bigvee_{k=1}^{d} (x_{ik} \wedge y_{ik}) \qquad (\mathbf{x},\mathbf{y} \in \{0,1\}^{dg})$$

  which is again AND-OR-AND RMM! $(d' = g^2, d'' = d)$

Plan



Solving AND-OR-AND RMM: The "Polynomial Method"

- rewrite $f(\mathbf{x}, \mathbf{y})$ as a multivariate polynomial
- problem is then reduced to standard RMM where
  new dim. $=$ # monomials of f
- Ex: $f(\mathbf{x}, \mathbf{y}) = x_1 y_2 + 4 x_2 y_1 y_2 + 3 x_1 x_2 y_1$
  $= (x_1, 4x_2, 3x_1 x_2) \cdot (y_2, y_1 y_2, y_1)$

Solving AND-OR-AND RMM: The "Polynomial Method"

- New Problem: rewrite

$$\text{AND-OR-AND}(\mathbf{x}, \mathbf{y}) = \bigwedge_{i=1}^{d'} \bigvee_{j=1}^{d''} (x_{ij} \wedge y_{ij})$$

as a polynomial with small # monomials

Solving AND-OR-AND RMM: The "Polynomial Method"

- New Problem: rewrite

$$\text{AND-OR}(\mathbf{z}) = \bigwedge_{i=1}^{d'} \bigvee_{j=1}^{d''} z_{ij}$$

  as a polynomial with small # monomials

- aim for small degree...

Razborov–Smolensky's OR Trick

Q: how to write a polynomial for $\bigvee_{j=1}^{d''} z_j$?

Easy Sol'n: $1 - \prod_{j=1}^{d''} (1 - z_j)$ (but degree $= d''$, too big!)

Razborov–Smolensky's OR Trick

Q: how to write a polynomial for $\bigvee_{j=1}^{d''} z_j$?

Randomized Sol'n:

- take random vector $\mathbf{r} \in \{0,1\}^{d''}$
- return $\sum_{j=1}^{d''} r_j z_j \pmod 2$

Analysis: degree $= 1$!

- false $\Rightarrow$ always correct
- true $\Rightarrow$ error prob. $= 1/2$
- can lower error prob. to $1/d'$ by repeating $\approx \log d'$ times & taking product $\Rightarrow$ degree $\approx \log d'$

Randomized AND-OR Polynomial

- just apply Razborov–Smolensky twice
  (for the top AND, use de Morgan & const error prob.)
- degree $\approx \log d'$
- # monomials $\approx d' \cdot \binom{d''}{\log d'}$

$$\leq O(\frac{d''}{\log d'})^{\log d'}$$

$$= 2^{O(\log^2 d)} \quad \text{for APSP} \quad (d', d'' = \text{poly}(d))$$

$$\ll n^{0.17}$$

by setting $d = \boxed{2^{\Theta(\sqrt{\log n})}}$

Derandomizing Razborov–Smolensky

- $2^{d''}$ choices for random vector $\mathbf{r}$
- Idea 1: use a smaller sample space that behaves roughly the same

  $\Rightarrow$ ε-biased sets

  [Naor–Naor'93, Alon–Goldreich–Hastad–Peralta'92, . . .]

Fact 1: $\exists$ set $\mathcal{R}_\varepsilon$ of poly$(d'', 1/\varepsilon)$ vectors s.t. if $\bigvee_{j=1}^{d''} z_j$ is true, then

$$\Pr_{\mathbf{r}\in\mathcal{R}_\varepsilon}\left[\sum_{j=1}^{d''} r_j z_j \equiv 1 \bmod 2\right] \in (1/2-\varepsilon, 1/2+\varepsilon)$$

Derandomizing Razborov–Smolensky

- now try all random choices $\mathbf{r} \in \mathcal{R}_\varepsilon \ldots$
  but can't tally the results when working mod 2!
- Idea 2: use a larger modulus

  $\Rightarrow$ modulus-amplifying polynomials
  [Yao'90, Beigel–Tarui'94, on ACC circuits, . . .]

Fact 2: $\exists$ polynomial F_ℓ of degree $O(\ell)$ s.t.

$$x \equiv 0 \bmod 2 \;\Rightarrow\; F_\ell(x) \equiv 0 \bmod 2^\ell$$
$$x \equiv 1 \bmod 2 \;\Rightarrow\; F_\ell(x) \equiv 1 \bmod 2^\ell$$

Final Deterministic AND-OR Polynomial

Rewrite AND-OR(z) $= \bigwedge_{i=1}^{d'} \bigvee_{j=1}^{d''} z_{ij}$ as

$$P(\mathbf{z}) := \sum_{i=1}^{d'} \sum_{\mathbf{r} \in \mathcal{R}_\varepsilon} F_\ell \left(\sum_{j=1}^{d''} r_j z_{ij} \right)$$

Final Deterministic AND-OR Polynomial

Rewrite AND-OR$(\mathbf{z}) = \bigwedge_{i=1}^{d'} \bigvee_{j=1}^{d''} z_{ij}$ as

$$P(\mathbf{z}) := \sum_{i=1}^{d'} \sum_{\mathbf{r} \in \mathcal{R}_\varepsilon} F_\ell \left(\sum_{j=1}^{d''} r_j z_{ij} \right)$$

- true $\Rightarrow$ $P(\mathbf{z}) \approx d' \, |\mathcal{R}_\varepsilon| \, (1/2 \pm \varepsilon) \mod 2^\ell$
- false $\Rightarrow$ $P(\mathbf{z}) \leq (d'-1) \, |\mathcal{R}_\varepsilon| \, (1/2 \pm \varepsilon) \mod 2^\ell$
- set $2^\ell \approx d'|\mathcal{R}_\varepsilon|$, $\varepsilon \approx 1/d'$ (to distinguish true vs. false)
- $|\mathcal{R}_\varepsilon| = \text{poly}(d'', 1/\varepsilon) = \text{poly}(d', d'')$
- degree $= O(\ell) = O(\log |\mathcal{R}_\varepsilon|) = O(\log(d'd''))$
- so, # monomials basically same as randomized Q.E.D.

More Applications

- also works for SUM-OR instead of AND-OR
  $\Rightarrow$ can count # orthogonal pairs of vectors

- offline dominance, orthogonal range searching, or exact L_∞ nearest neighbor search in $n^2/2^{\Omega(\sqrt{\log n})}$ det. time in dim. $d \leq 2^{O(\sqrt{\log n})}$

- can solve #k-SAT in $2^{(1-1/O(k))n}$ det. time