On Levels in Arrangements of Curves, III:

Further Improvements

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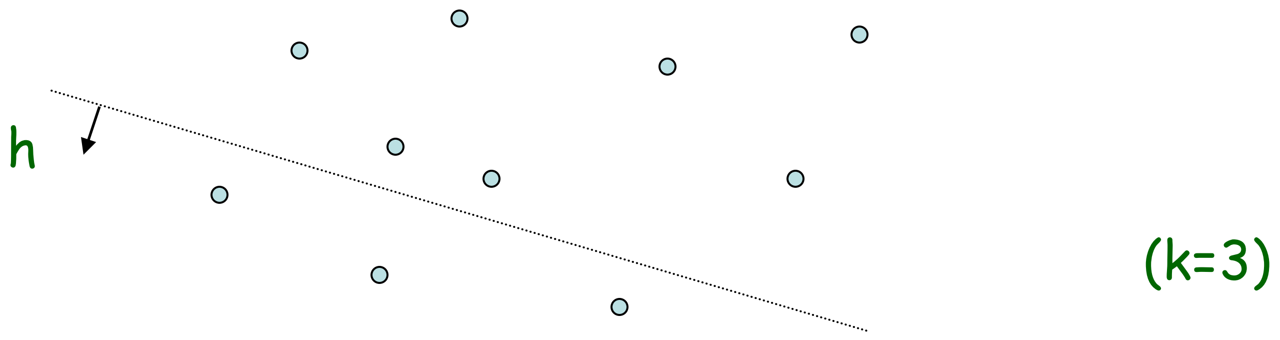
Back Story

- [FOCS'00] "On Levels in Arrangements of Curves"
- [FOCS'03] "On Levels in Arrangements of Curves, II"

- [SODA'05] " ... of Surfaces in 3D"
- [SODA'08] "On the Bichromatic k-Set Problem"

The k-Set Problem

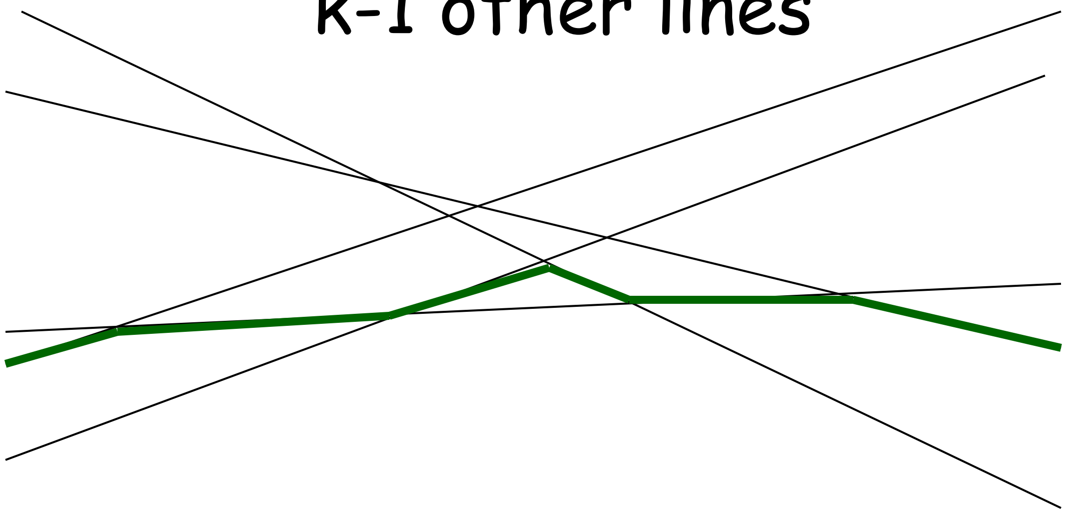
- Given n pts P in 2D & k, a k-set is a subset $P \cap h$ of size k for some halfplane h



- How many k-sets possible (as fn of n)?

Equiv. Dual Problem: k-Level of Lines

- Given n lines in 2D & k,
  k-level := all pts that lie on 1 line & above k-1 other lines



(k=2)

- How many vertices on k-level?

Known Upper Bds

- Lovász'71 $O(n^{3/2})$
- Gusfield'79 $O(n^{3/2})$
- Edelsbrunner,Welzl'85 $O(n^{3/2})$
- Pach,Steiger,Szemerédi'89 $O(n^{3/2} / \log^* n)$
- Dey'97 $O(n^{4/3})$

  (by counting crossings in a geometric graph in primal space)

Known Lower Bds

- Erdős,Lovász,Simmons,Straus'73 — $\Omega(n \log n)$
- Klawe,Paterson,Pippenger'82 — $n2^{\Omega(\sqrt{\log n})}$

  (for k-level of pseudo-lines)
- Tóth'00 — $n2^{\Omega(\sqrt{\log n})}$

- Conjecture by Erdős et al.'73:

  $o(n^{1+\varepsilon})$??

Going "Retro"

- Chan [FOCS'03]
  - a completely different pf of $O(n^{3/2})$
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Going "Retro"

- Chan [FOCS'03]
  - a completely different pf of $O(n^{3/2})$

    (by a simple inequality in dual space)

- Today:
  - a new pf of $O(n^{3/2 - \delta})$ for a concrete but small $\delta > 0$

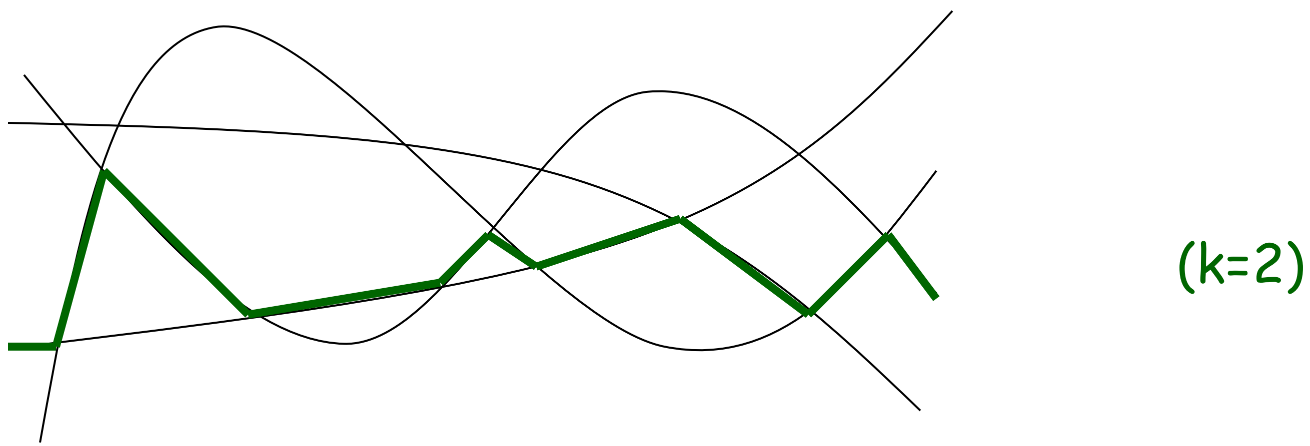
    (by refining the inequality)

Why Interesting, Despite of Dey?

- Because the inequality approach
  - is "completely different"
  - is the only approach known for k-level of general s-intersecting curves [FOCS'03]
  - gives current best bds for k-level of specific curve families [FOCS'03] & for related problems (e.g., k-level of surfaces in 3D [SODA'05], "bichromatic" k-sets in 2D & 3D [SODA'08])

The k-Level Problem for Curves

- Given n x-monotone curves in 2D & k,
  k-level := all pts that lie on 1 curve & above k-1 other curves



- How many vertices on k-level?

Known Upper Bds

- Pseudo-lines:

Dey + Tamaki,Tokuyama'97	$O(n^{4/3})$

- Pseudo-parabolas:

Tamaki,Tokuyama'95	$O(n^{23/12})$
Dey'97	$O(n^{17/9})$
[FOCS'00]	$O(n^{16/9} \log^{2/3} n)$
Agarwal,Nevo,Pach,Pinchasi, Sharir,Smorodinsky'02	$O(n^{26/15} \log^{2/3} n)$
[FOCS'03]	$O(n^{8/5})$
Marcus,Tardos'04	$O(n^{3/2} \log^2 n)$
Today	$O(n^{3/2} \log n)$

Known Upper Bds (cont'd)

- Pseudo-segments:

[FOCS'00] $O(n^{4/3} \log^{2/3} n)$

Today $O(n^{4/3} \log^{1/3-\delta} n)$

- Degree-s polynomials:

[FOCS'00] $O(n^{2 - O(1/2^s)})$

- General s-intersecting curves/curve segments:

[FOCS'03] $O(n^{2 - 1/(2s)})$ for s odd

$O(n^{2 - 1/(2s-2)})$ for $s \geq 4$ even

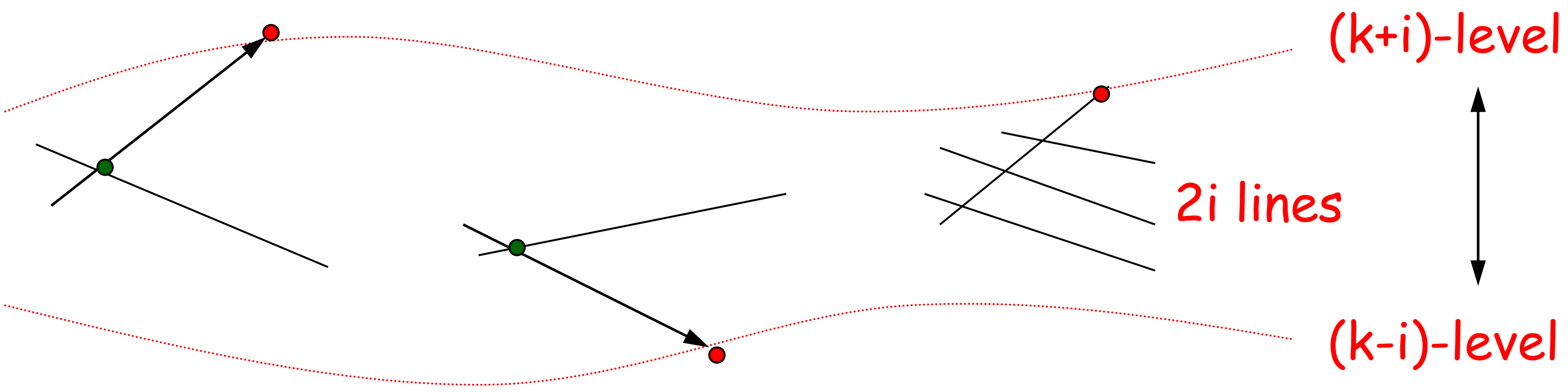
Today $O(n^{2 - 1/(2s) - \delta_s})$ for s odd

$O(n^{2 - 1/(2s-2) - \delta_s})$ for $s \geq 4$ even

The Inequality Approach [FOCS'03]: The (Pseudo-)Line Case

- Let t_i = # vertices at levels in interval (k-i,k+i)
- Let $\Delta t_i = t_{i+1} - t_i$

- Obs: $t_i \lesssim 2i\,\Delta t_i$
- Pf:



The Inequality Approach (cont'd)

- Solve $t_i \lesssim 2i\,(t_{i+1} - t_i)$

$\Rightarrow \quad t_i \lesssim [2i/(2i+1)]\, t_{i+1}$

$\Rightarrow \quad t_1 \lesssim 2/3 \cdot 4/5 \cdot 6/7 \cdots [2n/(2n+1)]\; t_n$

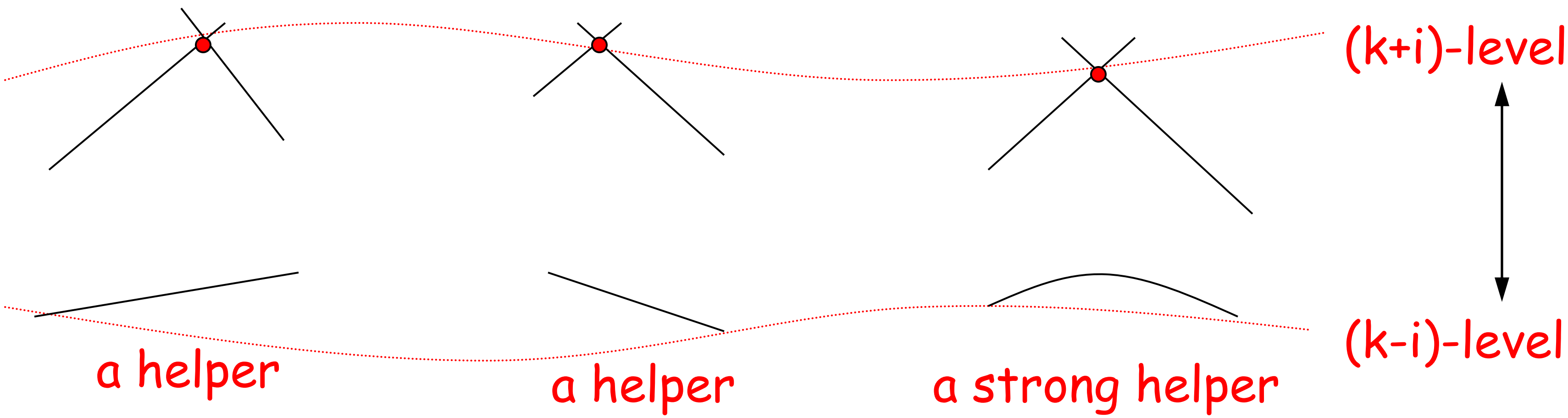
$\leq O(1/n^{1/2})\; t_n$

$= O(n^{3/2})$

- Rmk: $O(n^{3/2} + B)$ for "almost" pseudo-lines/segments with B "bad" pairs

New Idea 1: Look for Slack

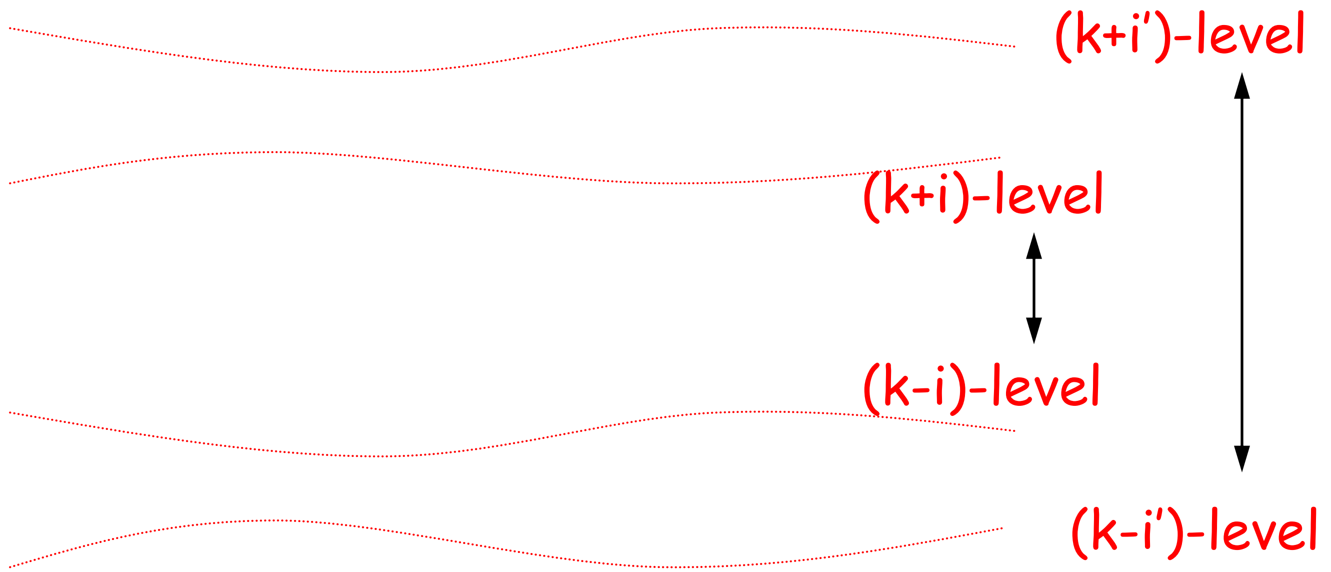
- Define helpers:



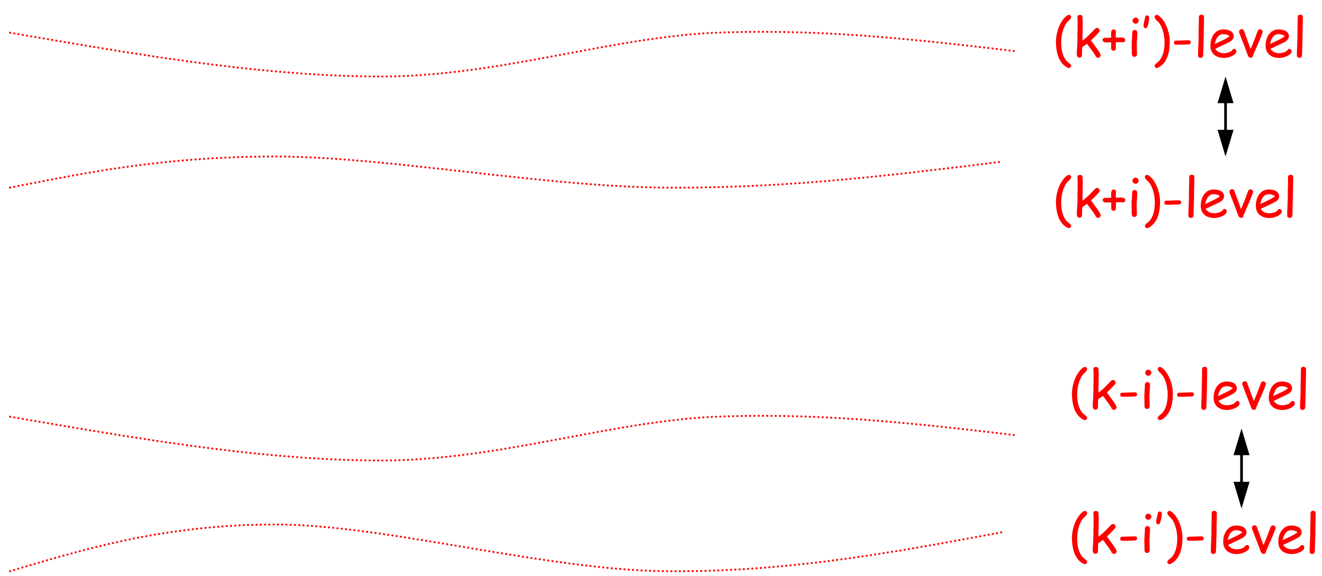
- Obs: $t_i \lesssim 2i\,\Delta t_i - \Omega(\#\text{ helpers})$

- But how to find helpers?

New Idea 2: Look at Two Intervals



New Idea 3: Look at The "Side" Intervals



Obs: $t_{i'} - t_i \gtrsim (i'-i)/2\ (\Delta t_i + \Delta t_{i'})$

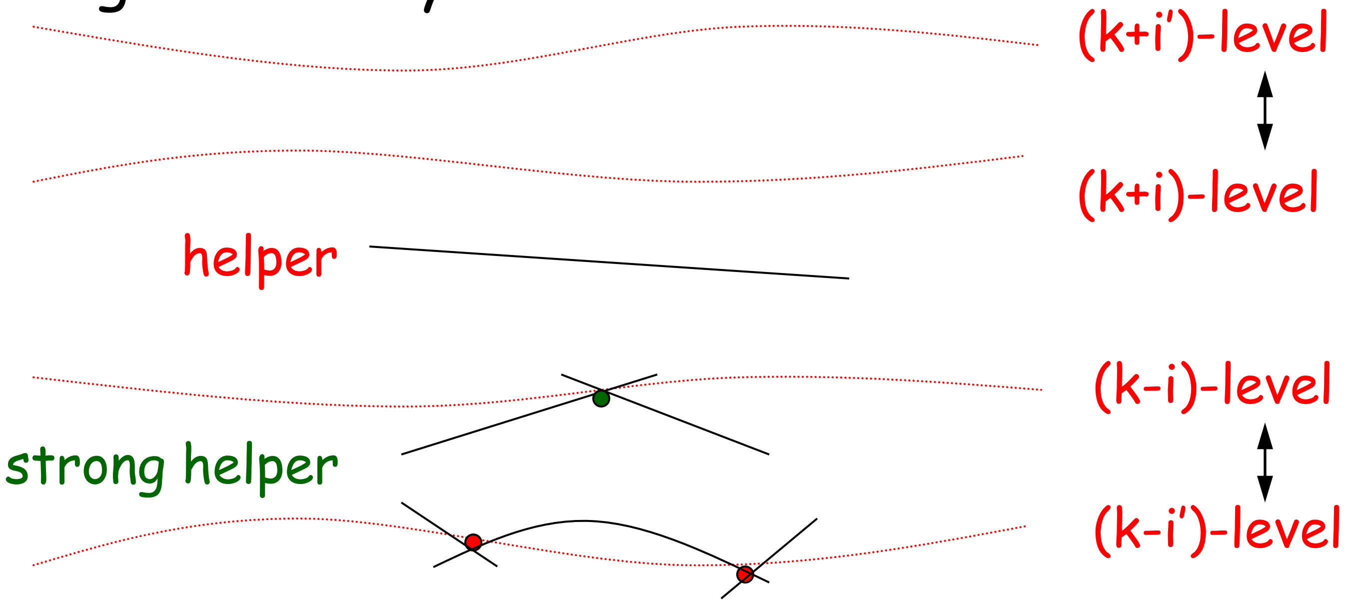
$-$ O(# strong helpers in side intervals)

New Idea 4: Surprise!

- Lemma: # helpers in the two intervals
  $\gtrsim$ c · # strong helpers in side intervals

- Pf: Long case analysis...

  e.g.,



New Inequality

$$t_i + t_{i'} \lesssim 2i\,\Delta t_i + 2i\,\Delta t_{i'} - c\,[(i'-i)/2\,(\Delta t_i + \Delta t_{i'}) - (t_{i'} - t_i)]$$

- Solve

$$\Rightarrow\ t_i \leq O(1/n^{1/2+\delta})\,t_n = O(n^{3/2-\delta})$$

Further Results/Open Questions

- Another, simpler way to refine the inequality for s-intersecting curves for odd $s \geq 3$ (but not curve segments or even s)
- Original k-set problem: $O(n^{4/3 - \delta})$??
- Another observation:
  - Better bds for const n $\Rightarrow$ better asymptotic upper bds
- k-level of curves: better upper or lower bds ??