Combinatorial Geometry & Approximation Algorithms

Timothy Chan
U. of Waterloo
PROLOGUE

Analysis of Approx Factor in
Analysis of Runtime in
Computational Geometry

Combinatorial Geometry
Problem 1: Geometric Set Cover

- Given $m$ points $P$ & $n$ (weighted) objects $S$, find min(-weight) subset of objects that cover all points
Problem 1’: Geometric Dual Set Cover
(i.e. Hitting Set/Piercing)

• Given $m$ objects $S$ & $n$ (weighted) points $P$, find min(-weight) subset of points that hit all objects

[continuous case: $P =$ entire space (unwt’ed)]
Problem 2: Geometric Indep Set
(or Set Packing)

• Given \(m\) points \(P\) & \(n\) (weighted) objects \(S\), find max(-weight) subset of objects s.t. no 2 chosen objects contain a common point.

[continuous case: \(P = \text{entire space}\)]
Problem 2’: Geometric Dual Indep Set

• Given $m$ objects $S$ & $n$ (weighted) points $P$, find max(-weight) subset of points s.t. no 2 chosen points are in a common object
History 1: Approx Set Cover

- **General:**
  - \text{wt'ed} \quad \ln m \quad (\text{greedy/LP})

- **2D (pseudo-)disks, 2D fat rectangles, 3D unit cubes, 3D halfspaces:**
  - \text{unwt'ed} \quad \mathcal{O}(1) \quad \text{Brönnimann, Goodrich, SoCG'94 /}
    \text{Clarkson, Varadarajan, SoCG'05 (LP)}
  - \text{wt'ed} \quad 2^{\mathcal{O}(\log^* n)} \quad \text{Varadarajan, STOC'10 (LP)}
  - \text{ptas} \quad \mathcal{O}(1) \quad \text{C., Grant, Könemann, Sharpe, SODA'12 (LP)}

- **2D disks, 3D halfspaces:**
  - \text{unwt'ed} \quad \text{PTAS} \quad \text{Mustafa, Ray, SoCG'09 (local search)}
### History 2: Approx Set Cover

**• 2D fat triangles:**

<table>
<thead>
<tr>
<th></th>
<th>Unweighted</th>
<th>Weighted</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$O(\log \log n)$</td>
<td>$O(\log \log \log n)$</td>
</tr>
<tr>
<td></td>
<td>Aronov, Ezra, Sharir, STOC'09 / Varadarajan, SoCG'09 (LP)</td>
<td>Aronov, de Berg, Ezra, Sharir, SODA'11 (LP)</td>
</tr>
<tr>
<td></td>
<td>$O(\log \log^* n)$</td>
<td>$2^O(\log^* n)$</td>
</tr>
<tr>
<td></td>
<td>Varadarajan, STOC'10 (LP)</td>
<td>C., Grant, Könemann, Sharpe, SODA'12 (LP)</td>
</tr>
</tbody>
</table>
Continuous case:

- **dD unit balls, unit hypercubes:**
  - unwt'ed PTAS Hochbaum, Maass'85 (shifted grid+DP)
- **dD balls, hypercubes, general fat objects:**
  - unwt'ed PTAS C.'03 (separator)
- **2D unit-height rectangles:**
  - unwt'ed PTAS C., Mahmood'05 (shifted grid+DP)
History 4: Approx Dual Set Cover

Discrete case:

- 2D unit disks, 3D unit cubes, 3D halfspaces:
  
<table>
<thead>
<tr>
<th>Unw'ted</th>
<th>Wt'ted</th>
<th>Authors/Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(1)$</td>
<td>$2^{O(\log^* n)}$</td>
<td>Brönnimann,Goodrich,SoCG'94 (LP)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$O(1)$</th>
<th>Varadarajan,STOC’10 (LP)</th>
</tr>
</thead>
</table>

- 2D (pseudo-)disks, 3D halfspaces:
  
<table>
<thead>
<tr>
<th>Unw'ted</th>
<th>Wt'ted</th>
<th>Authors/Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>PTAS</td>
<td>$2^{O(\log^* n)}$</td>
<td>Mustafa, Ray, SoCG'09 (local search)</td>
</tr>
</tbody>
</table>

- 2D rectangles, 3D boxes:
  
<table>
<thead>
<tr>
<th>Unw'ted</th>
<th>Wt'ted</th>
<th>Authors/Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(\log \log n)$</td>
<td>Aronov,Ezra,Sharir,STOC'09 (LP)</td>
<td></td>
</tr>
</tbody>
</table>
History 5: Approx Indep Set

Continuous case:

- dD unit balls, unit hypercubes:
  - wt'ed PTAS Hochbaum,Maass'85 (shifted grid+DP)
- 2D unit-height rectangles:
  - wt'ed PTAS Agarwal,van Kreveld,Suri'97 (shifted grid+DP)
- dD balls, hypercubes, general fat objects:
  - wt'ed PTAS Erlebach,Jansen,Seidel,SODA'01 / C.'03 (shifted quadtree+DP)
- 2D pseudo-disks:
  - unwt'ed PTAS C.,Har-Peled,SoCG'09 (local search)
  - wt'ed $O(1)$ C.,Har-Peled,SoCG'09 (LP)
Continuous case:

- **2D rectangles:**
  - wt'ed $\log n$
  - $\delta \log n$  
  - $O(\log n / \log \log n)$

  - Agarwal, van Kreveld, Suri'97 (D&C)
  - Berman, DasGupta, Muthukrishnan, Ramaswami, SODA'01 (D&C+DP)
  - C., Har-Peled, SoCG'09 (LP)
  - Chalermsook, Chuzhoy, SODA'09 (LP)

- **dD boxes:**
  - wt'ed $O((\log n)^{d-2} / \log \log n)$
  - unwt'ed $O(((\log n)^{d-1} \log \log n)$

  - C., Har-Peled, SoCG'09 (LP)
  - Chalermsook, Chuzhoy, SODA'09 (LP)

- **2D line segments:**
  - unwt'ed $\tilde{O}(\sqrt{n})$  
  - $O(n^\delta)$

  - Agarwal, Mustafa'04
  - Fox, Pach, SODA'11 (separator)
History 7: Approx Indep Set

Discrete case:

- 2D (pseudo-)disks, 2D fat rectangles:
  wt'ed \( O(1) \) C., Har-Peled, SoCG'09 (LP)

- 2D disks, 3D halfspaces:
  unwt'ed PTAS Ene, Har-Peled, Raichel, SoCG'12 (local search)

- 2D fat triangles:
  wt'ed \( O(\log^* n) \) C., Har-Peled, SoCG'09 (LP)

- dD boxes:
  wt'ed \( O(\log n) \) in 2D Ene, Har-Peled, Raichel, SoCG'12 (D&C)
  \( O((\log n)^3) \) in 3D
  \( O(n^{1-0.632/(2^{2d-3}-0.368)}) \) C., SoCG'12 (LP)
History 8: Approx Dual Indep Set

- 2D (pseudo-)disks, 3D halfspaces:
  - unwt'ed PTAS Ene, Har-Peled, Raichel, SoCG'12 (local search)
- dD boxes:
  - wt'ed
    - $O(n^{0.368})$ in 2D C., SoCG'12 (LP)
    - $O(n^{1 - 0.632/2^{d-2}})$
PART I

Approx Set Cover

LP rounding

$\varepsilon$-Nets

[Varadarajan, STOC'10 /
C., Grant, Könemann, Sharpe, SODA'12]

Union Complexity

$(\leq k)$-Level Complexity
Problem: $\varepsilon$-Nets

- Given $n$ (weighted) objects, an $\varepsilon$-net is a subset of objects that covers all points of level $\geq \varepsilon n$
  where level of $p = \#$ objects containing $p$
- Prove that $\exists$ $\varepsilon$-net of small size (or weight)
  (as function of $\varepsilon$)
## History: $\varepsilon$-Nets

- **General:**
  
  $O((1/\varepsilon) \log m)$

- **Bounded VC dim:**
  
  $O((1/\varepsilon) \log(1/\varepsilon))$  
  Vapnik, Chervonenkis'71 / Haussler, Welzl, SoCG'86

  $O\left(W/n \cdot (1/\varepsilon) \log(1/\varepsilon)\right)$

- **2D (pseudo-)disks, 2D fat rectangles, 3D unit cubes, 3D halfspaces:**
  
  $O(1/\varepsilon)$  
  Matousek, Seidel, Welzl, SoCG'90

  $O\left(W/n \cdot (1/\varepsilon) \cdot 2^{O(\log^*(1/\varepsilon))} \right)$  
  Varadarajan, STOC'10

  $O\left(W/n \cdot (1/\varepsilon)\right)$  
  C., Grant, Könemann, Sharpe, SODA'12
History: $\varepsilon$-Nets

- **2D fat triangles:**
  \[ O\left(\frac{1}{\varepsilon} \log \log(1/\varepsilon)\right) \]
  Clarkson, Varadarajan, SoCG'05
  \[ O\left(\frac{1}{\varepsilon} \log \log \log(1/\varepsilon)\right) \]
  Aronov, Ezra, Sharir, STOC'09 / Varadarajan, SoCG'09
  \[ O\left(\frac{1}{\varepsilon} \log \log^*(1/\varepsilon)\right) \]
  Aronov, de Berg, Ezra, Sharir, SODA'11
  \[ O\left(\frac{W}{n} \cdot (1/\varepsilon)^{2O(\log^*(1/\varepsilon))}\right) \]
  Varadarajan, STOC'10
  \[ O\left(\frac{W}{n} \cdot (1/\varepsilon) \log \log^*(1/\varepsilon)\right) \]
  C., Grant, Könemann, Sharpe, SODA'12

- **2D dual rectangles, 3D dual boxes:**
  \[ O\left(\frac{1}{\varepsilon} \log \log(1/\varepsilon)\right) \]
  Aronov, Ezra, Sharir, STOC'09
### $\varepsilon$-Nets → Approx Set Cover

[Brönnimann, Goodrich, SoCG'94 / Even, Rawitz, Shahar'05]

- Assume unwt'ed case & $\varepsilon$-net complexity $O((1/\varepsilon) f(1/\varepsilon))$

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Solve LP: $\min \sum_{\text{object } s} x_s$</td>
<td></td>
</tr>
<tr>
<td>s.t. $\sum_{s \text{ contains } p} x_s \geq 1 \quad \forall \text{ point } p$</td>
<td></td>
</tr>
<tr>
<td>$0 \leq x_s \leq 1$</td>
<td></td>
</tr>
<tr>
<td>2. Let $S'$ be multiset where each obj $s$ is duplicated $[Mx_s]$ times</td>
<td></td>
</tr>
<tr>
<td>3. Return $\varepsilon$-net $R$ of $S'$</td>
<td></td>
</tr>
</tbody>
</table>

- $|S'| \approx \sum_s Mx_s = M \text{ OPT}_{LP}$
- $\forall p$, level of $p$ in $S' \approx \sum_{s \text{ contains } p} Mx_s \geq M$
- $\Rightarrow$ can set $\varepsilon \approx 1/\text{OPT}_{LP}$
- $\Rightarrow |R| = O(\text{OPT}_{LP} f(\text{OPT}_{LP})) \leq O(\text{OPT} f(\text{OPT}))$
Problem: Union Complexity

• Given $n$ objects, prove that boundary of the union has small \# vertices (as function of $n$)
Problem: Union Complexity

• Given $n$ objects, prove that boundary of the union has small # vertices (as function of $n$)
History: Union Complexity

- **3D halfspaces:**
  \[ O(n) \] by planar graph

- **2D (pseudo-)disks, 2D fat rectangles:**
  \[ O(n) \] Kedem, Livne, Pach, Sharir '86

- **3D unit cubes:**
  \[ O(n) \] Boissonnat, Sharir, Tagansky, Yvinec, SoCG '95

- **2D fat triangles:**
  \[ O(n \log \log n) \] Matoušek, Pach, Sharir, Sifrony, Welzl, FOCS '91
  \[ O(n \log^* n) \] Aronov, de Berg, Ezra, Sharir, SODA '11

- **Etc., etc., etc.**
Problem: ($\leq k$)-Level Complexity

• Given $n$ objects & given $k$, prove that the arrangement has small # vertices/cells of level $\leq k$
  (as function of $n$ & $k$)
Union Complexity $\rightarrow (\leq k)$-Level
[Clarkson,Shor'88]

- Assume 2D & union complexity $O(n f(n))$
- Take random sample $R$ where each obj is picked w. prob $1/k$
- $\forall$ vertex $v$ of level $\leq k$,
  \[ \Pr[v \text{ is on boundary of union of } R] \geq \left(\frac{1}{k}\right)^2 \left(1 - \frac{1}{k}\right)^k = \Omega(1/k^2) \]
  \[ \Rightarrow \]
  \[ O((n/k) f(n/k)) \geq E[\# \text{ vertices on boundary of union of } R] \geq \Omega(1/k^2) \cdot [\# \text{ vertices of level } \leq k] \]
  \[ \Rightarrow (\leq k)$-level complexity $O(nk f(n/k)) \]
(≤ k)-Level → ε-Nets
[Varadarajan,STOC’10 / C.,Grant,Könemann,Sharpe,SODA'12]

• Assume (≤ k)-level complexity $O(nk f(n/k))$ with $f(\cdot) = O(1)$

Definition: A $\rho$-sample $R$ of $S$ is a subset where
    each object is picked w. prob $\rho$ (independently)

Definition: A quasi-$\rho$-sample $R$ of $S$ is a subset s.t.
    $\forall$ object $s$, $\Pr[s \in R] = O(\rho)$
    (but events $\{s \in R\}$ may not be independent!)
Lemma: Let \( R \) be \((1/2 + c\sqrt{(\log k)/k})\)-sample of \( S \)

Then \( p \) has level \( \geq k \) in \( S \)

\[ \Rightarrow p \text{ has level } \geq k/2 \text{ in } R \quad \text{w. prob } 1 - O(1/k^{102}) \]

Proof:

- \( E[\text{level of } p \text{ in } R] \geq k \cdot (1/2 + c\sqrt{(\log k)/k}) \)

- Use Chernoff bound \quad \text{Q.E.D.}
(≤ k)-Level → ε-Nets

[Varadarajan,STOC’10 / C.,Grant,Könemann,Sharpe,SODA'12]

“Correction” Lemma: Let $R$ be $(1/2 + c \sqrt{\log k/k})$-sample of $S$

Then $\exists$ quasi-$O(1/k^{100})$-sample $A$ of $S$ s.t.

$p$ has level $\geq k$ in $S$

$\Rightarrow$ $p$ has level $\geq k/2$ in $R$ or $p$ is covered by $A$

Proof:

• # cells of level $\leq k$ is $O(nk)$

• Each such cell is contained in $\leq k$ objects

$\Rightarrow$ $\exists$ “low-degree” object $s$ that contains $O(k^2)$ cells of level $\leq k$

• Inductively handle $S - \{s\}$

• If $s$ contains a cell that has level $k$ in $S$ but level $< k/2$ in $R$,

  then add $s$ to $A$

$\Rightarrow$ $\Pr[s \in A] \leq O(k^2 \cdot 1/k^{102})$ Q.E.D.
\((\leq k)\)-Level \(\rightarrow\) \(\varepsilon\)-Nets

[Varadarajan,STOC’10 / C.,Grant,Könemann,Sharpe,SODA'12]

**Corollary (after \(\ell\) iterations):** Let \(R\) be \((\approx 1/2^{\ell})\)-sample of \(S\)

Then \(\exists\) quasi-\(\rho\)-sample \(A\) of \(S\) with

\[\rho \approx \sum_{i=0}^{\ell-1} \frac{1}{(k/2^i)^{100}} \cdot \frac{1}{2^i}\]

s.t.

\(p\) has level \(\geq k\) in \(S\)

\(\Rightarrow\) \(p\) has level \(\geq k/2^{\ell}\) in \(R\) or \(p\) is covered by \(A\)

- Set \(k = \varepsilon n\), \(\ell = \log k\), & return \(R \cup A\)

\(\Rightarrow\) \(\rho = O(1/k)\) by geometric series

\(\Rightarrow E[|R \cup A|] = O(n/k) = O(1/\varepsilon)\)

[in general, \(O((1/\varepsilon) \log f(1/\varepsilon)))\)]
PART I (Recap)

Approx Set Cover

LP rounding

\(\varepsilon\)-Nets

[Varadarajan, STOC’10 / C., Grant, Königemann, Sharpe, SODA'12]

Union Complexity \(\leq k\)-Level Complexity

Complexity
PART II

Approx Indep Set

LP rounding
[C., Har-Peled, SoCG'09]

Union Complexity

(≤ k)-Level Complexity
$(\leq k)$-Level $\rightarrow$ Approx Indep Set

[C., Har-Peled, SoCG'09]

- Assume unwt'ed, continuous case

1. Solve LP: $\max \ \sum_{\text{object } s} y_s$
   
   s.t. $\sum_{s \text{ contains } p} y_s \leq 1 \ \forall \ \text{point } p$

2. Let $R$ be random sample where object $s$ is picked w. prob $y_s$

3. Return indep set $Q$ in intersect. graph of $R$ by Turan's theorem
\( (\leq k) \)-Level \( \rightarrow \) Approx Indep Set

[C., Har-Peled, SoCG'09]

- **Turan's Theorem**: Any graph with \( n \) vertices & average degree \( D \) has indep set of size \( \geq \frac{n}{D + 1} \)
(≤ k)-Level → Approx Indep Set

[C., Har-Peled, SoCG'09]

• Assume (≤ k)-level complexity \( O(nk f(n/k)) \)

• Let \( S' \) be multiset where each object \( s \) is duplicated \([My_s]\) times

• \( |S'| \approx \sum_s My_s = M \\text{OPT}_{LP} \)

• ∀\( p \), level of \( p \) in \( S' \) \( \approx \sum_s \text{contains}_p My_s \leq M \)

\[ \Rightarrow \]

\[ \sum_{s,t \text{ intersect}} My_s My_t \approx \]

# vertices in arrangement of \( S' \) = \( O(M \\text{OPT}_{LP} M f(\text{OPT}_{LP})) \)
\((\leq k)\)-Level \(\rightarrow\) Approx Indep Set

[C., Har-Peled, SoCG'09]

- Assume \((\leq k)\)-level complexity \(O(nk f(n/k))\)

- Let \(S'\) be multiset where each object \(s\) is duplicated \([My_s]\) times

- \(|S'| \approx \sum_s My_s = M \text{ OPT}_{LP}\)

- \(\forall p, \text{ level of } p \in S' \approx \sum_s \text{ contains}_p My_s \leq M\)

\[\Rightarrow \]

\[\sum_{s,t \text{ intersect}} My_s My_t \approx\]

\# vertices in arrangement of \(S' = O(M \text{ OPT}_{LP} M f(\text{OPT}_{LP}))\)
\((\leq k)\)-Level \(\rightarrow\) Approx Indep Set

[C., Har-Peled, SoCG'09]

- Assume unwt'ed, continuous case

1. Solve LP: \(\max \sum_{\text{objects } s} y_s\)
   s.t. \(\sum_{s \text{ contains } p} y_s \leq 1 \quad \forall \text{ point } p\)
   \(0 \leq y_s \leq 1\)

2. Let \(R\) be random sample where object \(s\) is picked w. prob \(y_s\)

3. Return indep set \(Q\) in intersect. graph of \(R\) by Turan's theorem

- \(E[|R|] = \sum_s y_s = \text{OPT}_{\text{LP}}\)
- \(E[\# \text{ intersect. pairs of } R] = \sum_{s,t \text{ intersect}} y_s y_t\)
  \(= O(\text{OPT}_{\text{LP}} f(\text{OPT}_{\text{LP}}))\)

\(\Rightarrow\) average degree in intersect. graph of \(R\) is \(O(f(\text{OPT}_{\text{LP}}))\)

\(\Rightarrow\) \(E[|Q|] \geq \Omega(\text{OPT}_{\text{LP}} / f(\text{OPT}_{\text{LP}})) \geq \Omega(\text{OPT} / f(\text{OPT}))\)
PART II (Alternate)

Approx Indep Set

LP rounding
[C.,SoCG’12]

Conflict-Free Coloring ⇄ Indep Set in Delaunay Graphs
Problem: Conflict-Free (CF) Coloring

• Given \( n \) objects, prove that we can color them with small \# colors (as function of \( n \)) s.t.

\[ \forall \text{ point } p \text{ of level } \geq 1, \text{ there is a unique color among the objects containing } p \]
History: CF Coloring

- **2D (pseudo-)disks, 3D halfspaces:**
  \[ O(\log n) \]
  Even, Lotker, Ron, Smorodinsky, FOCS'02 / Har-Peled, Smorodinsky, SoCG'03

- **2D fat triangles:**
  \[ O(\log n \log^* n) \]
  Aronov, de Berg, Ezra, Sharir, SODA'11

- **2D rectangles:**
  \[ O((\log n)^2) \]
  Har-Peled, Smorodinsky, SoCG'03
History: CF Coloring

- **2D dual rectangles:**
  \[ O(\sqrt{n}) \quad \text{Har-Peled,Smorodinsky,SoCG'03} \]
  \[ O(\sqrt{n/\log n}) \quad \text{Pach,Tardos/Alon/...'03} \]
  \[ O(n^{0.382}) \quad \text{Ajwani,Elbassioni,Govindarajan,Ray'07} \]
  \[ O(n^{0.368}) \quad \text{C.,SoCG'12} \]

- **dD dual boxes:**
  \[ O(n^{1-0.632/2^{d-2}}) \quad \text{C.,SoCG'12} \]

- **dD boxes:**
  \[ O(n^{1-0.632/(2^{2d-3}-0.368)}) \quad \text{C.,SoCG'12} \]
Problem: Indep Set in Delaunay Graph

- Given $n$ objects, the Delaunay graph (DG) has an edge between objects $s, t$ iff $\exists$ point $p$ that is in both $s, t$ & has level $2 \leq k$
- Prove that $\exists$ indep set in DG of large size
  (as function of $n$)
CF Coloring $\leftrightarrow$ Indep Set in DG

[Even, Lotker, Ron, Smorodinsky, FOCS'02 / Har-Peled, Smorodinsky, SoCG'03]

$(\rightarrow)$ Assume CF coloring with $\Theta(f(n))$ colors

$\Rightarrow$ largest color class is an indep set in DG of size $\Omega(n/f(n))$

$(\leftarrow)$ Assume indep set in DG of size $\Omega(n/f(n))$

Make it a new color class, remove, repeat

$\Rightarrow$ CF coloring with $\tilde{O}(f(n))$ colors [under certain conditions]
Indep Set in DG $\rightarrow$ Approx Indep Set 
[C.,SoCG'12]

- Assume unwt'ed case & indep set size $\Omega(n/f(n))$ in DG

1. Solve LP: $\max \sum_{s} \text{object}_s y_s$
   
   s.t. $\sum_{s \text{contains } p} y_s \leq 1 \ \ \forall \ \text{point } p$
   
   $0 \leq y_s \leq 1$

2. let $R$ be random sample where object $s$ is picked w. prob $y_s$

3. return indep set $Q$ in order-$k$ DG of $R$

- $\forall p$, $E[\text{level of } p \text{ in } R] = \sum_{s \text{contains } p} y_s \leq 1$

  $\Rightarrow$ can set $k \approx \log n$

- $E[|R|] = \sum_{s} y_s = \text{OPT}_{\text{LP}}$

  $\Rightarrow$ w.h.p., $|Q| \geq \tilde{\Omega}(\text{OPT}_{\text{LP}}/f(\text{OPT}_{\text{LP}})) \geq \tilde{\Omega}(\text{OPT} / f(\text{OPT}))$
EPILOGUE

Computational Geometry

\[\uparrow\]

Combinatorialial Geometry
Example

[Wegner’67]

• Given $n$ (unwt’ed) rectangles in 2D,
  let $\text{OPT}_{\text{hit}} = \min \# \text{ points that hit all rectangles}$
  $\text{OPT}_{\text{indep}} = \max \# \text{ disjoint rectangles}$

• Prove that $\frac{\text{OPT}_{\text{hit}}}{\text{OPT}_{\text{indep}}}$ is small (as function of $n$)
Theorem: \( \text{OPT}_{\text{hit}} / \text{OPT}_{\text{indep}} \leq O((\log \log n)^2) \)

Proof:

- Aronov, Ezra, Sharir, STOC'09 \( \Rightarrow \) \( \text{OPT}_{\text{hit}} \leq O(\log \log n) \) \( \text{OPT}_{\text{LP}}: \)

\[
\begin{align*}
\min & \sum_{\text{point } p} x_p \\
\text{s.t.} & \sum_{p \text{ in } s} x_p \geq 1 \quad \forall \text{ rectangle } s \\
& 0 \leq x_p \leq 1 
\end{align*}
\]

- Chalermsook, Chuzhoy, SODA'09 \( \Rightarrow \) \( \text{OPT}_{\text{indep}} \geq \text{OPT}_{\text{LP}} / O(\log \log n) \): \( \text{OPT}_{\text{indep}} \geq \text{OPT}_{\text{LP}} / O(\log \log n) \):

\[
\begin{align*}
\max & \sum_{\text{rectangle } s} y_s \\
\text{s.t.} & \sum_{s \text{ contains } p} y_s \leq 1 \quad \forall \text{ point } p \\
& 0 \leq y_s \leq 1 
\end{align*}
\]

- But the 2 LPs are dual! \( \text{Q.E.D.} \)