Combinatorial Geometry & Approximation Algorithms

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PROLOGUE

Analysis of Approx Factor in

~~Analysis of Runtime in~~

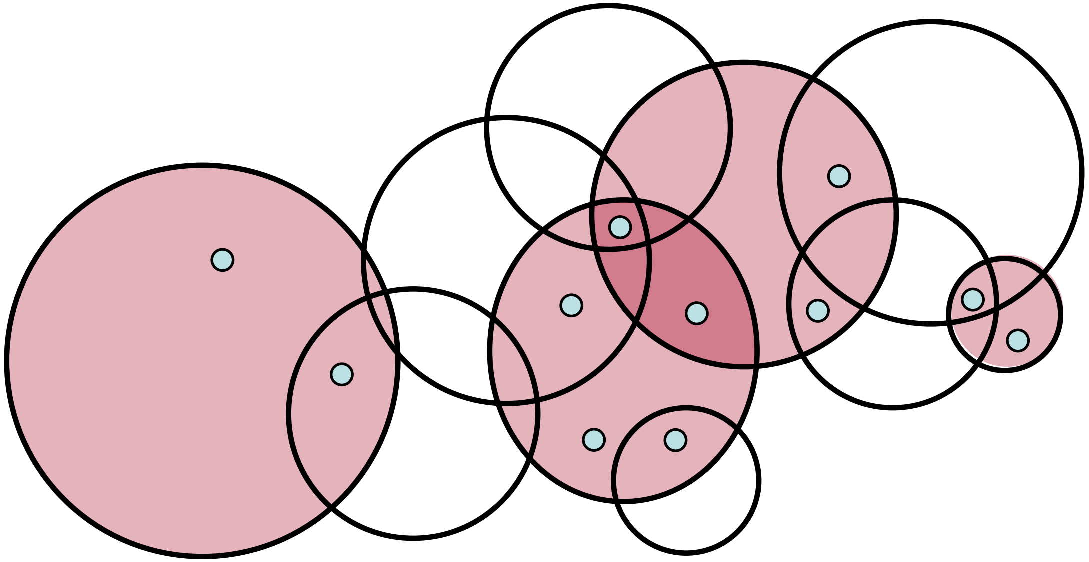
Computational Geometry

↑

Combinatorial Geometry

Problem 1: Geometric Set Cover

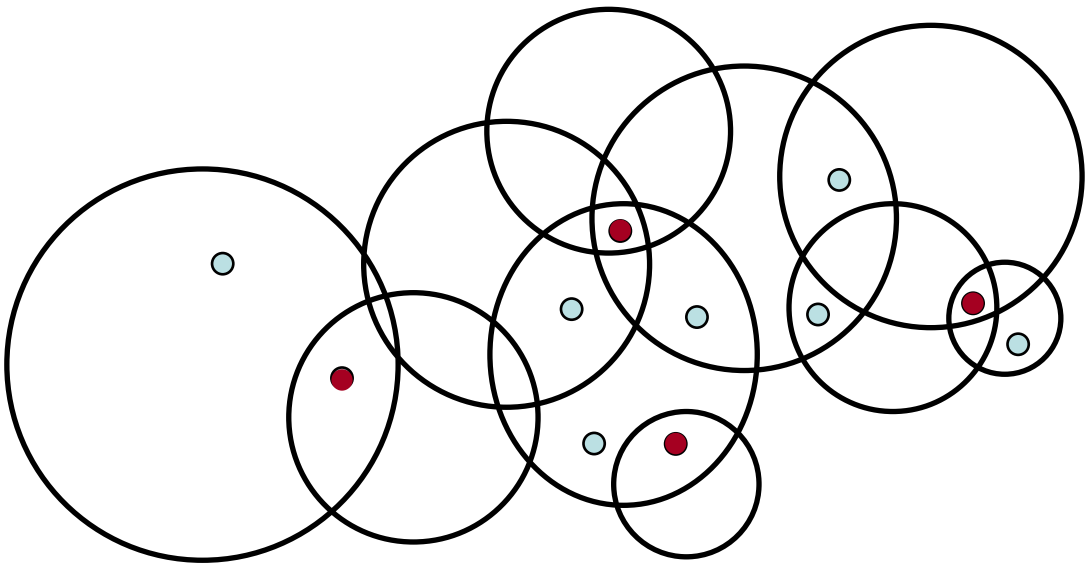
- Given m points P & n (weighted) objects S, find min(-weight) subset of objects that cover all points



Problem 1’: Geometric Dual Set Cover

(i.e. Hitting Set/Piercing)

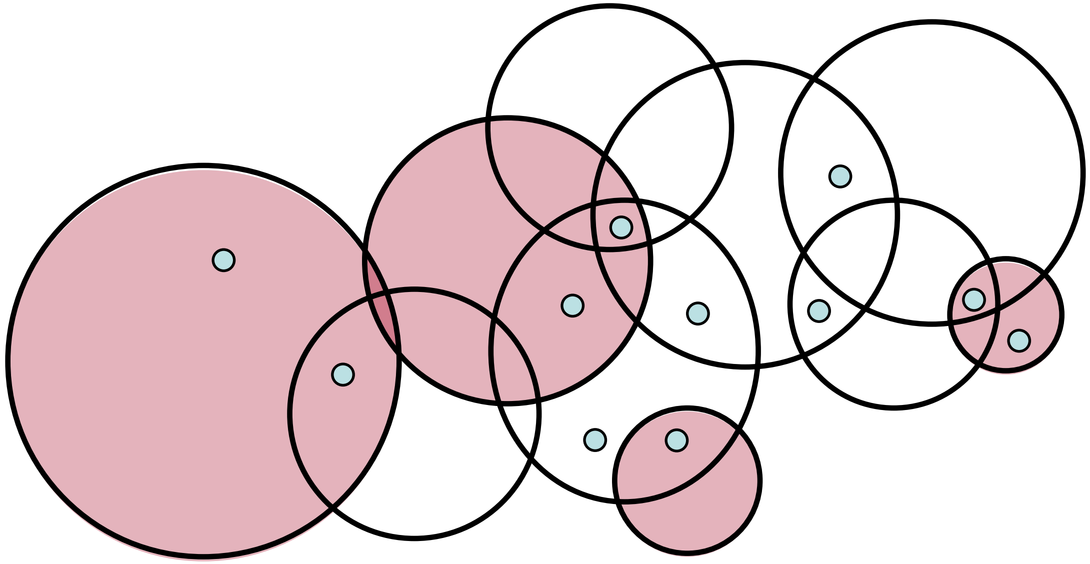
- Given m objects S & n (weighted) points P, find min(-weight) subset of points that hit all objects



[continuous case: $P =$ entire space (unwt’ed)]

Problem 2: Geometric Indep Set
(or Set Packing)

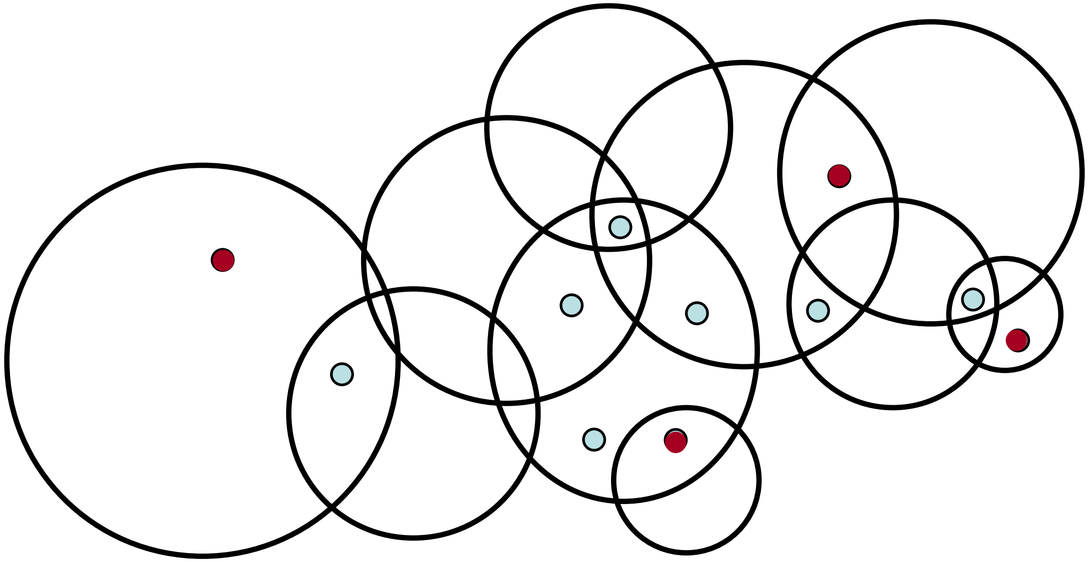
- Given m points P & n (weighted) objects S,
  find max(-weight) subset of objects
  s.t. no 2 chosen objects contain a common point



[continuous case: P = entire space]

Problem 2': Geometric Dual Indep Set

- Given m objects S & n (weighted) points P,
  find max(-weight) subset of points
  s.t. no 2 chosen points are in a common object



History 1: Approx Set Cover

- General:

wt'ed	$\ln m$	(greedy/LP)

- 2D (pseudo-)disks, 2D fat rectangles, 3D unit cubes, 3D halfspaces:

unwt'ed	$O(1)$	Brönnimann,Goodrich,SoCG'94 / Clarkson,Varadarajan,SoCG'05 (LP)
wt'ed	$2^{O(\log^* n)}$	Varadarajan,STOC'10 (LP)
	$O(1)$	C.,Grant,Könemann,Sharpe,SODA'12 (LP)

- 2D disks, 3D halfspaces:

unwt'ed	PTAS	Mustafa,Ray,SoCG'09 (local search)

History 2: Approx Set Cover

- 2D fat triangles:

unwt'ed	$O(\log\log n)$	Clarkson,Varadarajan,SoCG'05 (LP)
	$O(\log\log\log n)$	Aronov,Ezra,Sharir,STOC'09 / Varadarajan,SoCG'09 (LP)
	$O(\log\log^* n)$	Aronov,de Berg,Ezra,Sharir,SODA'11 (LP)
wt'ed	$2^{O(\log^* n)}$	Varadarajan,STOC'10 (LP)
	$O(\log\log^* n)$	C.,Grant,Könemann,Sharpe,SODA'12 (LP)

History 3: Approx Dual Set Cover

Continuous case:

- dD unit balls, unit hypercubes:

  unwt'ed PTAS Hochbaum,Maass'85 (shifted grid+DP)

- dD balls, hypercubes, general fat objects:

  unwt'ed PTAS C.'03 (separator)

- 2D unit-height rectangles:

  unwt'ed PTAS C.,Mahmood'05 (shifted grid+DP)

History 4: Approx Dual Set Cover

Discrete case:

- 2D unit disks, 3D unit cubes, 3D halfspaces:

unwt'ed	$O(1)$	Brönnimann,Goodrich,SoCG'94 (LP)
wt'ed	$2^{O(\log^* n)}$	Varadarajan,STOC'10 (LP)
	$O(1)$	C.,Grant,Könemann,Sharpe,SODA'12 (LP)

- 2D (pseudo-)disks, 3D halfspaces:

unwt'ed	PTAS	Mustafa,Ray,SoCG'09 (local search)

- 2D rectangles, 3D boxes:

unwt'ed	$O(\log \log n)$	Aronov,Ezra,Sharir,STOC'09 (LP)

History 5: Approx Indep Set

Continuous case:

- dD unit balls, unit hypercubes:

wt'ed	PTAS	Hochbaum,Maass'85 (shifted grid+DP)

- 2D unit-height rectangles:

wt'ed	PTAS	Agarwal,van Kreveld,Suri'97 (shifted grid+DP)

- dD balls, hypercubes, general fat objects:

wt'ed	PTAS	Erlebach,Jansen,Seidel,SODA'01 / C.'03 (shifted quadtree+DP)

- 2D pseudo-disks:

unwt'ed	PTAS	C.,Har-Peled,SoCG'09 (local search)
wt'ed	O(1)	C.,Har-Peled,SoCG'09 (LP)

History 6: Approx Indep Set

Continuous case:

- 2D rectangles:

wt'ed	$\log n$	Agarwal,van Kreveld,Suri'97 (D&C)
	$\delta \log n$	Berman,DasGupta,Muthukrishnan, Ramaswami,SODA'01 (D&C+DP)
	$O(\log n / \log \log n)$	C.,Har-Peled,SoCG'09 (LP)
unwt'ed	$O(\log \log n)$	Chalermsook,Chuzhoy,SODA'09 (LP)

- dD boxes:

wt'ed	$O((\log n)^{d-2} / \log \log n)$	C.,Har-Peled,SoCG'09 (LP)
unwt'ed	$O(((\log n)^{d-1} \log \log n)$	Chalermsook,Chuzhoy,SODA'09 (LP)

- 2D line segments:

unwt'ed	$\tilde{O}(\sqrt{n})$	Agarwal,Mustafa'04
	$O(n^{\delta})$	Fox,Pach,SODA'11 (separator)

History 7: Approx Indep Set

Discrete case:

- 2D (pseudo-)disks, 2D fat rectangles:

wt'ed	$O(1)$	C.,Har-Peled,SoCG'09 (LP)

- 2D disks, 3D halfspaces:

unwt'ed	PTAS	Ene,Har-Peled,Raichel,SoCG'12 (local search)

- 2D fat triangles:

wt'ed	$O(\log^* n)$	C.,Har-Peled,SoCG'09 (LP)

- dD boxes:

wt'ed	$O(\log n)$ in 2D	Ene,Har-Peled,Raichel,SoCG'12 (D&C)
	$O((\log n)^3)$ in 3D	
	$O(n^{1-0.632/(2^{2d-3}-0.368)})$	C.,SoCG'12 (LP)

History 8: Approx Dual Indep Set

- 2D (pseudo-)disks, 3D halfspaces:

unwt'ed PTAS Ene,Har-Peled,Raichel,SoCG'12 (local search)

- dD boxes:

wt'ed

$O(n^{0.368})$ in 2D	C.,SoCG'12 (LP)
$O(n^{1-0.632/2^{d-2}})$	

PART I

Approx Set Cover

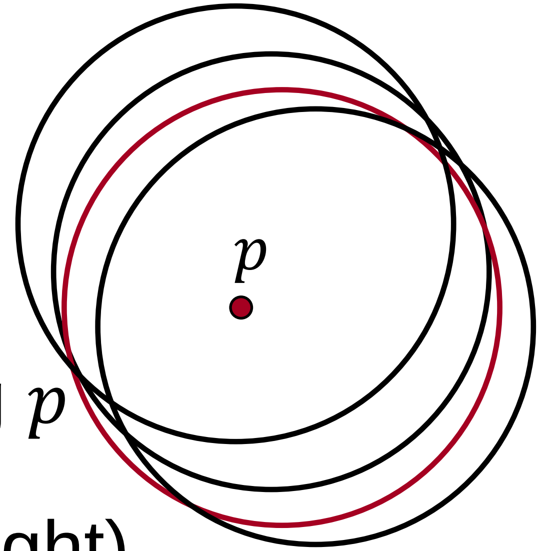
LP rounding

ε-Nets

[Varadarajan,STOC'10 /
C.,Grant,Könemann,Sharpe,SODA'12]

Union Complexity $\longleftrightarrow$ $(\leq k)$-Level Complexity

Problem: ε-Nets



- Given n (weighted) objects,
  an ε-net is a subset of objects that
  covers all points of level $\geq \varepsilon n$

where level of $p = \#$ objects containing p

- Prove that $\exists$ ε-net of small size (or weight)
  (as function of ε)

History: ε-Nets

- General:

$O((1/\varepsilon)\log m)$

- Bounded VC dim:

$O((1/\varepsilon)\log(1/\varepsilon))$ — Vapnik,Chervonenkis'71 / Haussler,Welzl,SoCG'86

$O(W/n \cdot (1/\varepsilon)\log(1/\varepsilon))$

- 2D (pseudo-)disks, 2D fat rectangles, 3D unit cubes, 3D halfspaces:

$O(1/\varepsilon)$ — Matousek,Seidel,Welzl,SoCG'90 / Clarkson,Varadarajan,SoCG'05 / Pyrga,Ray,SoCG'08

$O(W/n \cdot (1/\varepsilon)2^{O(\log*(1/\varepsilon))})$ — Varadarajan,STOC’10

$O(W/n \cdot (1/\varepsilon))$ — C.,Grant,Könemann,Sharpe,SODA'12

History: ε-Nets

- 2D fat triangles:

$O((1/\varepsilon)\log\log(1/\varepsilon))$ Clarkson,Varadarajan,SoCG'05

$O((1/\varepsilon)\log\log\log(1/\varepsilon))$ Aronov,Ezra,Sharir,STOC'09 / Varadarajan,SoCG'09

$O((1/\varepsilon)\log\log^*(1/\varepsilon))$ Aronov,de Berg,Ezra,Sharir,SODA'11

$O(W/n \cdot (1/\varepsilon)2^{O(\log*(1/\varepsilon))})$ Varadarajan,STOC'10

$O(W/n \cdot (1/\varepsilon)\log\log^*(1/\varepsilon))$ C.,Grant,Könemann,Sharpe,SODA'12

- 2D dual rectangles, 3D dual boxes:

$O((1/\varepsilon)\log\log(1/\varepsilon))$ Aronov,Ezra,Sharir,STOC'09

ε-Nets $\rightarrow$ Approx Set Cover

[Brönnimann,Goodrich,SoCG'94 / Even,Rawitz,Shahar'05]

- Assume unwt'ed case & ε-net complexity $O((1/\varepsilon)\, f(1/\varepsilon))$

1. Solve LP: min $\sum_{\text{object } s} x_s$

   s.t. $\sum_{s \text{ contains } p} x_s \geq 1 \quad \forall$ point p

   $0 \leq x_s \leq 1$
2. Let S' be multiset where each obj s is duplicated $\lceil M x_s \rceil$ times
3. Return ε-net R of S'

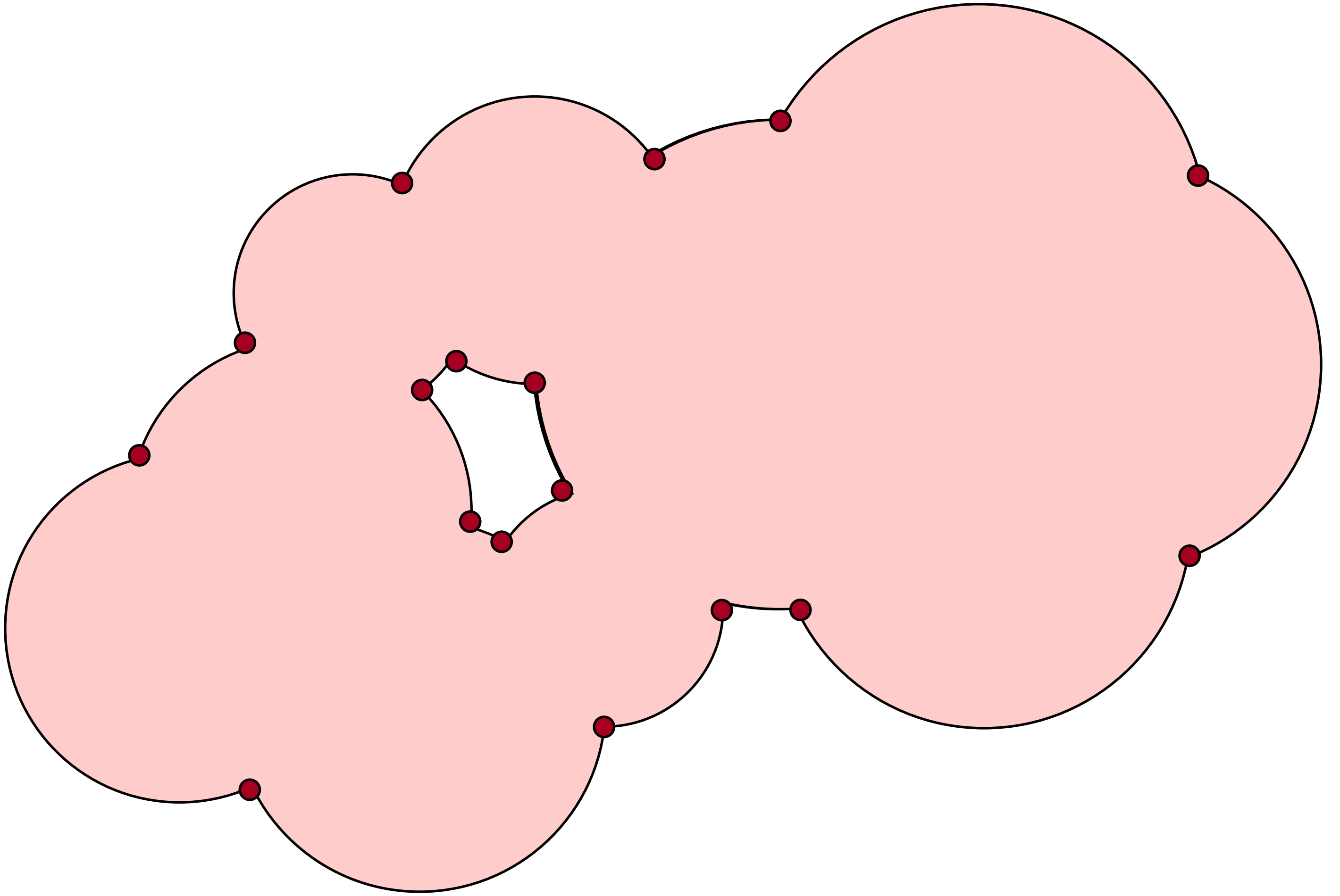
- $|S'| \approx \sum_s M x_s = M\,\text{OPT}_{\text{LP}}$
- $\forall p$, level of p in $S' \approx \sum_{s \text{ contains } p} M x_s \geq M$

$\Rightarrow$ can set $\varepsilon \approx 1/\text{OPT}_{\text{LP}}$

$\Rightarrow$ $|R| = O(\text{OPT}_{\text{LP}}\, f(\text{OPT}_{\text{LP}})) \leq O(\text{OPT}\, f(\text{OPT}))$

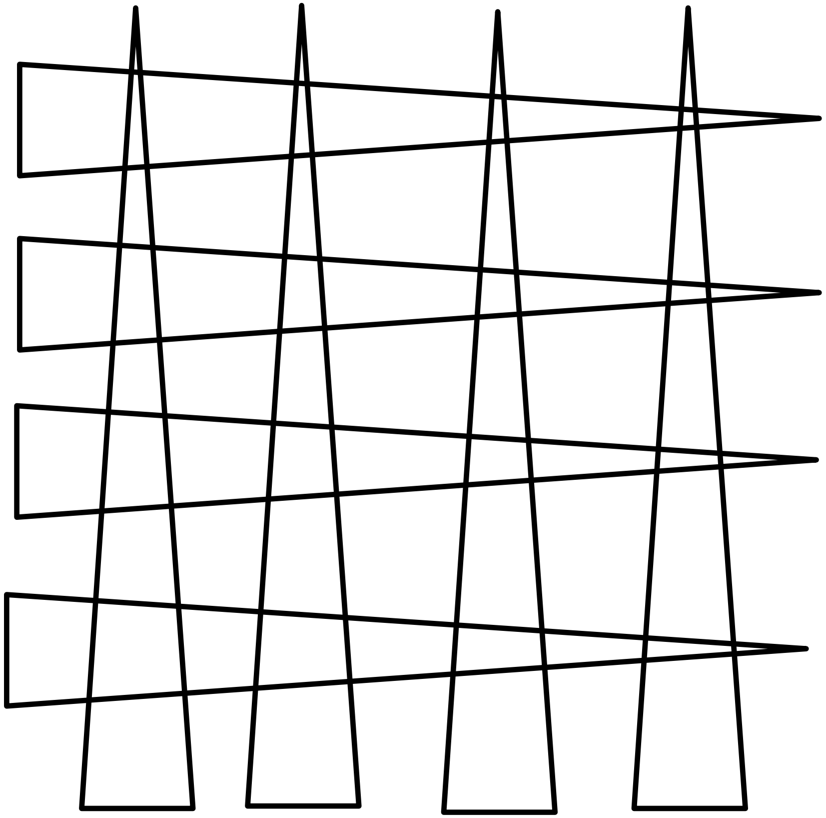
Problem: Union Complexity

- Given n objects, prove that boundary of the union has small # vertices (as function of n)



Problem: Union Complexity

- Given n objects, prove that boundary of the union has small # vertices (as function of n)



History: Union Complexity

- 3D halfspaces:

$O(n)$ by planar graph

- 2D (pseudo-)disks, 2D fat rectangles:

$O(n)$ Kedem,Livne,Pach,Sharir'86

- 3D unit cubes:

$O(n)$ Boissonnat,Sharir,Tagansky,Yvinec,SoCG'95

- 2D fat triangles:

$O(n \log \log n)$ Matoušek,Pach,Sharir,Sifrony,Welzl,FOCS'91

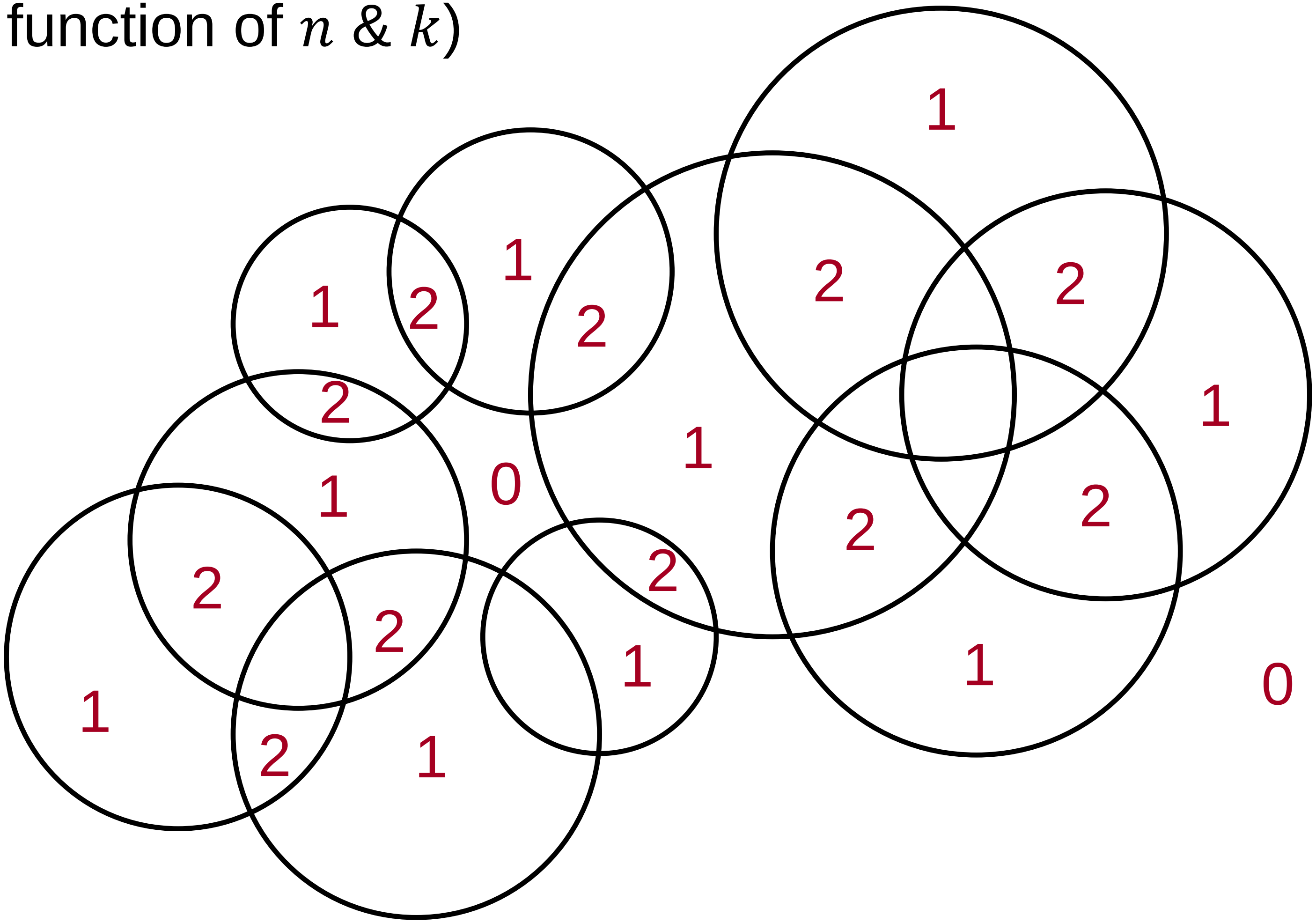
$O(n \log^* n)$ Aronov,de Berg,Ezra,Sharir,SODA'11

- Etc., etc., etc.

Problem: $(\leq k)$-Level Complexity

- Given n objects & given k, prove that the arrangement has small # vertices/cells of level $\leq k$

  (as function of n & k)



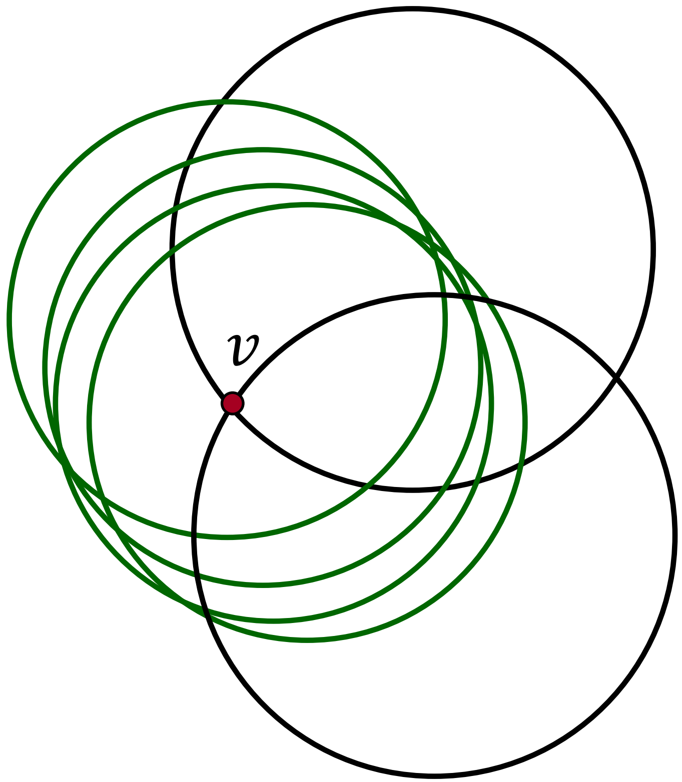
Union Complexity $\to$ $(\leq k)$-Level

[Clarkson,Shor'88]

- Assume 2D & union complexity $O(n\,f(n))$
- Take random sample R where each obj is picked w. prob $1/k$
- $\forall$ vertex v of level $\leq k$,

  $\Pr[v$ is on boundary of union of $R]$

  $\geq\ (1/k)^2\,(1-1/k)^k\ =\ \Omega(1/k^2)$



$\Rightarrow$

$O((n/k)\,f(n/k))\ \geq$

$E[$# vertices on boundary of union of $R]$

$\geq \Omega(1/k^2)\ \cdot$ [# vertices of level $\leq k$]

$\Rightarrow\ (\leq k)$-level complexity $O(nk\,f(n/k))$

$(\leq k)$-Level $\rightarrow$ ε-Nets

[Varadarajan,STOC'10 / C.,Grant,Könemann,Sharpe,SODA'12]

- Assume $(\leq k)$-level complexity $O(nk\, f(n/k))$ with $f(\cdot) = O(1)$

Definition: A ρ-sample R of S is a subset where
each object is picked w. prob ρ (independently)

Definition: A quasi-ρ-sample R of S is a subset s.t.
$\forall$ object s, $\Pr[s \in R] = O(\rho)$
(but events $\{s \in R\}$ may not be independent!)

$(\leq k)$-Level $\rightarrow$ ε-Nets

[Varadarajan,STOC'10 / C.,Grant,Könemann,Sharpe,SODA'12]

Lemma: Let R be $\left(1/2 + c\sqrt{(\log k)\ /k}\right)$-sample of S

Then p has level $\geq k$ in S

$\Rightarrow$ p has level $\geq k/2$ in R w. prob $1 - O(1/k^{102})$

Proof:

- E[level of p in R] $\geq k \cdot \left(1/2 + c\sqrt{(\log k)\ /k}\right)$
- Use Chernoff bound Q.E.D.

$(\leq k)$-Level $\rightarrow$ ε-Nets

[Varadarajan,STOC'10 / C.,Grant,Könemann,Sharpe,SODA'12]

"Correction" Lemma: Let R be $(1/2 + c\sqrt{(\log k)\,/k})$-sample of S

Then $\exists$ quasi-$O(1/k^{100})$-sample A of S s.t.

p has level $\geq k$ in S

$\Rightarrow$ p has level $\geq k/2$ in R or p is covered by A

Proof:

- # cells of level $\leq k$ is $O(nk)$
- Each such cell is contained in $\leq k$ objects

$\Rightarrow$ $\exists$ "low-degree" object s that contains $O(k^2)$ cells of level $\leq k$

- Inductively handle $S - \{s\}$
- If s contains a cell that has level k in S but level $< k/2$ in R,

  then add s to A

$\Rightarrow$ $\Pr[s \in A] \leq O(k^2 \cdot 1/k^{102})$ Q.E.D.

$(\leq k)$-Level $\rightarrow$ ε-Nets

[Varadarajan,STOC'10 / C.,Grant,Könemann,Sharpe,SODA'12]

Corollary (after ℓ iterations): Let R be ($\approx 1/2^{\ell}$)-sample of S

Then $\exists$ quasi-ρ-sample A of S with

$\rho \approx \sum_{i=0}^{\ell-1} 1/(k/2^i)^{100} \cdot 1/2^i$ s.t.

p has level $\geq k$ in S

$\Rightarrow$ p has level $\geq k/2^{\ell}$ in R or p is covered by A

- Set $k = \varepsilon n$, $\ell = \log k$, & return $R \cup A$

$\Rightarrow$ $\rho = O(1/k)$ by geometric series

$\Rightarrow$ $E[|R \cup A|] = O(n/k) = O(1/\varepsilon)$

[in general, $O((1/\varepsilon) \log f(1/\varepsilon))$]

PART I (Recap)

Approx Set Cover

↑ LP rounding

ε-Nets

↑ [Varadarajan,STOC’10 / C.,Grant,Könemann,Sharpe,SODA'12]

Union Complexity ⟷ $(\leq k)$-Level Complexity

PART II

Approx Indep Set

LP rounding

[C.,Har-Peled,SoCG'09]

Union Complexity $\longleftrightarrow$ $(\leq k)$-Level Complexity

$(\leq k)$-Level $\rightarrow$ Approx Indep Set

[C.,Har-Peled,SoCG'09]

- Assume unwt'ed, continuous case

1. Solve LP: max $\sum_{\text{object } s} y_s$

   s.t. $\sum_{s \text{ contains } p} y_s \leq 1 \quad \forall$ point p

   $0 \leq y_s \leq 1$
2. let R be random sample where object s is picked w. prob y_s
3. return indep set Q in intersect. graph of R by Turan's theorem

$(\leq k)$-Level $\rightarrow$ Approx Indep Set

[C.,Har-Peled,SoCG'09]

- Turan's Theorem: Any graph with n vertices & average degree D has indep set of size $\geq n/(D+1)$

$(\leq k)$-Level $\rightarrow$ Approx Indep Set

[C.,Har-Peled,SoCG'09]

- Assume $(\leq k)$-level complexity $O(nk\, f(n/k))$

- Let S' be multiset where each object s is duplicated $\lceil My_s \rceil$ times
- $|S'| \approx \sum_s My_s = M\ \mathrm{OPT}_{\mathrm{LP}}$
- $\forall p$, level of p in $S' \approx \sum_{s \text{ contains } p} My_s \leq M$

$\Rightarrow$

$\sum_{s,t \text{ intersect}} My_s\, My_t \approx$

vertices in arrangement of $S' = O(M\ \mathrm{OPT}_{\mathrm{LP}}\ M\, f(\mathrm{OPT}_{\mathrm{LP}}))$

$(\le k)$-Level $\to$ Approx Indep Set

[C.,Har-Peled,SoCG'09]

- Assume $(\le k)$-level complexity $O(nk\, f(n/k))$

- Let S' be multiset where each object s is duplicated $\lceil M y_s \rceil$ times
- $|S'| \approx \sum_s M y_s = M\ \mathrm{OPT}_{\mathrm{LP}}$
- $\forall p$, level of p in $S' \approx \sum_{s \text{ contains } p} M y_s \le M$

$\Rightarrow$

$$\sum_{s,t \text{ intersect}} M y_s\, M y_t \approx$$

$\#$ vertices in arrangement of $S' = O(M\ \mathrm{OPT}_{\mathrm{LP}}\ M\, f(\mathrm{OPT}_{\mathrm{LP}}))$

$(\le k)$-Level $\rightarrow$ Approx Indep Set

[C.,Har-Peled,SoCG'09]

- Assume unwt'ed, continuous case

1. Solve LP: max $\sum_{\text{object } s} y_s$

   s.t. $\sum_{s \text{ contains } p} y_s \le 1 \quad \forall$ point p

   $0 \le y_s \le 1$
2. let R be random sample where object s is picked w. prob y_s
3. return indep set Q in intersect. graph of R by Turan's theorem

- $E[|R|] = \sum_s y_s = \text{OPT}_{\text{LP}}$
- $E[\#$ intersect. pairs of $R] = \sum_{s,t \text{ intersect}} y_s y_t$

  $= O(\text{OPT}_{\text{LP}} f(\text{OPT}_{\text{LP}}))$

$\Rightarrow$ average degree in intersect. graph of R is $O(f(\text{OPT}_{\text{LP}}))$

$\Rightarrow$ $E[|Q|] \ge \Omega(\text{OPT}_{\text{LP}} / f(\text{OPT}_{\text{LP}})) \ge \Omega(\text{OPT} / f(\text{OPT}))$

PART II (Alternate)

Approx Indep Set

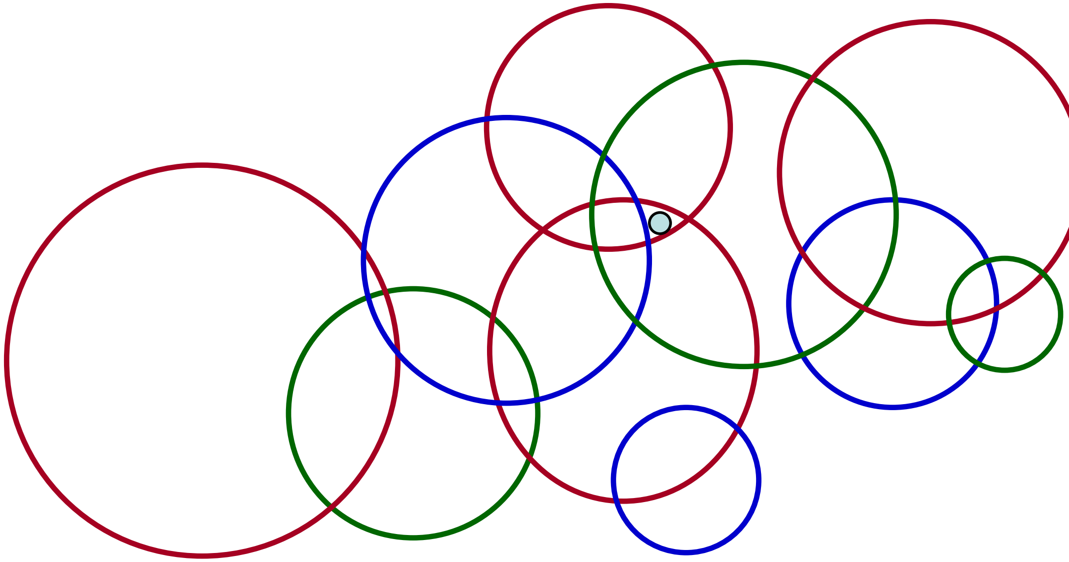
LP rounding
[C.,SoCG'12]

Conflict-Free Coloring ⟷ Indep Set in Delaunay Graphs

Problem: Conflict-Free (CF) Coloring

- Given n objects, prove that we can color them with small # colors (as function of n) s.t.

  $\forall$ point p of level ≥ 1, there is a unique color among the objects containing p



History: CF Coloring

- 2D (pseudo-)disks, 3D halfspaces:

$O(\log n)$ Even,Lotker,Ron,Smorodinsky,FOCS'02 /

Har-Peled,Smorodinsky,SoCG'03

- 2D fat triangles:

$O(\log n \ \log^* \ n)$ Aronov,de Berg,Ezra,Sharir,SODA'11

- 2D rectangles:

$O((\log n)^2)$ Har-Peled,Smorodinsky,SoCG'03

History: CF Coloring

- 2D dual rectangles:

$O(\sqrt{n})$	Har-Peled,Smorodinsky,SoCG'03
$O(\sqrt{n/\log n})$	Pach,Tardos/Alon/...'03
$O(n^{0.382})$	Ajwani,Elbassioni,Govindarajan,Ray'07
$O(n^{0.368})$	C.,SoCG'12

- dD dual boxes:

$O(n^{1-0.632/2^{d-2}})$	C.,SoCG'12

- dD boxes:

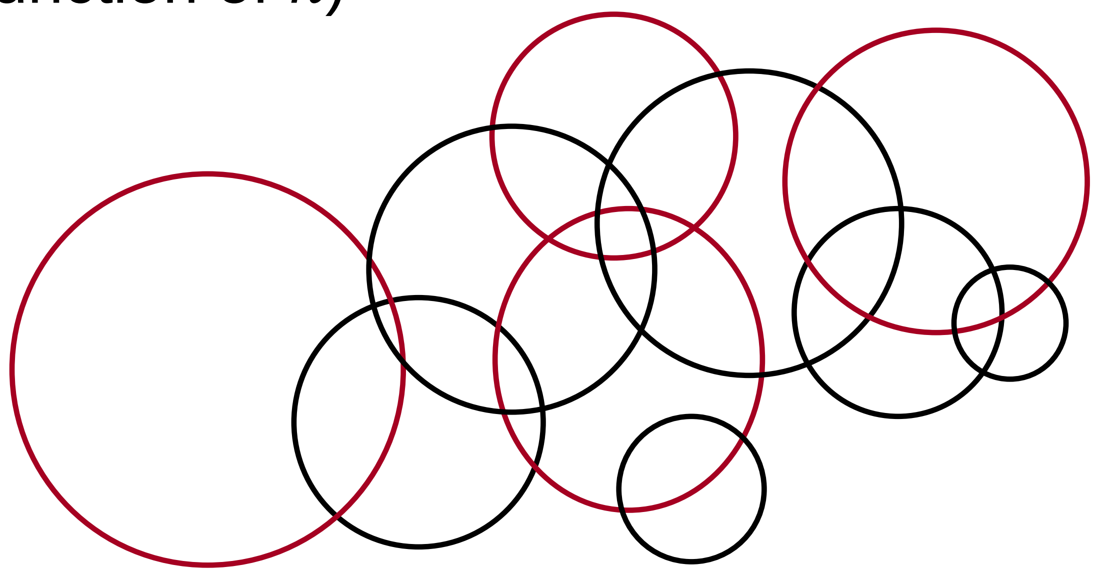
$O(n^{1-0.632/(2^{2d-3}-0.368)})$	C.,SoCG'12

Problem: Indep Set in Delaunay Graph

- Given n objects, the order-k Delaunay graph (DG) has an edge between objects s, t iff ∃ point p that is in both s, t & has level ~~2~~ $\leq k$

- Prove that ∃ indep set in DG of large size

  (as function of n)



CF Coloring $\leftrightarrow$ Indep Set in DG

[Even,Lotker,Ron,Smorodinsky,FOCS'02 / Har-Peled,Smorodinsky,SoCG'03]

($\rightarrow$) Assume CF coloring with $O(f(n))$ colors

$\Rightarrow$ largest color class is an indep set in DG of size $\Omega(n/f(n))$

($\leftarrow$) Assume indep set in DG of size $\Omega(n/f(n))$

Make it a new color class, remove, repeat

$\Rightarrow$ CF coloring with $\tilde{O}(f(n))$ colors [under certain conditions]

Indep Set in DG → Approx Indep Set

[C.,SoCG'12]

- Assume unwt'ed case & indep set size $\Omega(n/f(n))$ in DG

1. Solve LP: max $\sum_{\text{object } s} y_s$

   s.t. $\sum_{s \text{ contains } p} y_s \le 1 \quad \forall$ point p

   $0 \le y_s \le 1$
2. let R be random sample where object s is picked w. prob y_s
3. return indep set Q in order-k DG of R

- $\forall\, p$, $E[\text{level of } p \text{ in } R] = \sum_{s \text{ contains } p} y_s \le 1$

  $\Rightarrow$ can set $k \approx \log n$
- $E[|R|] = \sum_s y_s = \text{OPT}_{\text{LP}}$

  $\Rightarrow$ w.h.p., $|Q| \ge \tilde{\Omega}(\text{OPT}_{\text{LP}} / f(\text{OPT}_{\text{LP}})) \ge \tilde{\Omega}(\text{OPT} / f(\text{OPT}))$

EPILOGUE

Computational Geometry

⇅

Combinatorial Geometry

Example

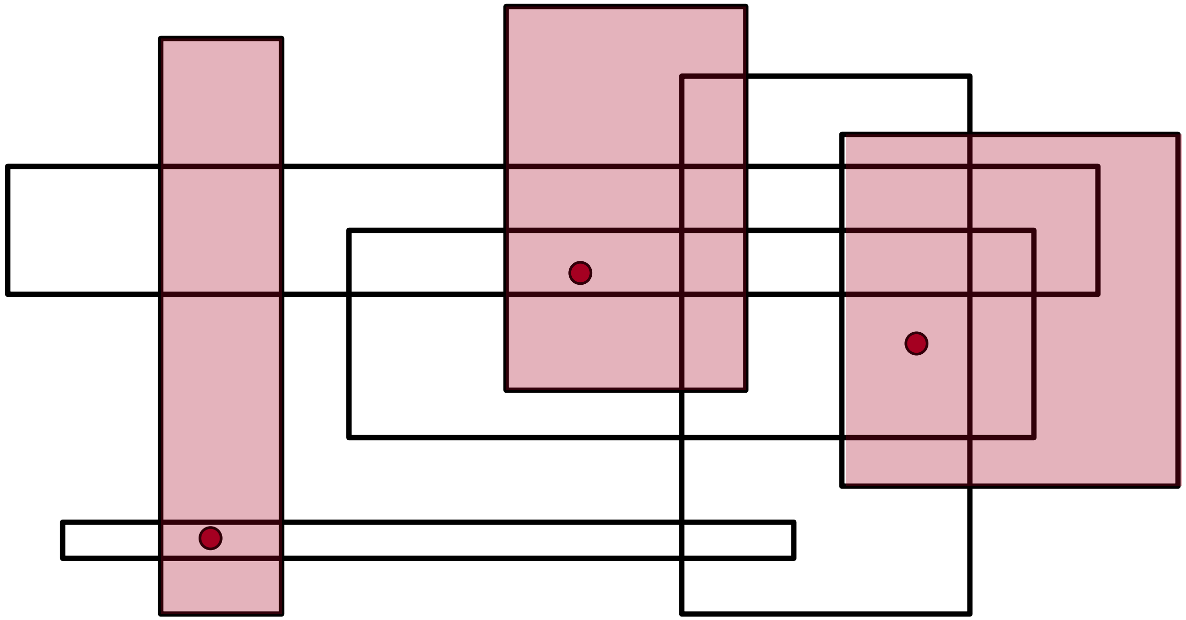
[Wegner'67]

- Given n (unwt'ed) rectangles in 2D,

  let OPT_{hit} = min # points that hit all rectangles

  $\text{OPT}_{\text{indep}}$ = max # disjoint rectangles

- Prove that OPT_{hit} / $\text{OPT}_{\text{indep}}$ is small (as function of n)



Example

Theorem: OPT_{hit} / $\text{OPT}_{\text{indep}} \leq O((\log\log n)^2)$

Proof:

- Aronov,Ezra,Sharir,STOC'09 $\Rightarrow$ $\text{OPT}_{\text{hit}} \leq O(\log\log n)\ \text{OPT}_{\text{LP}}$:

$$\begin{array}{ll} \min & \sum_{\text{point } p} x_p \\ \text{s.t.} & \sum_{p \text{ in } s} x_p \geq 1 \quad \forall \text{ rectangle } s \\ & 0 \leq x_p \leq 1 \end{array}$$

- Chalermsook,Chuzhoy,SODA'09 $\Rightarrow$ $\text{OPT}_{\text{indep}} \geq \text{OPT}_{\text{LP}} / O(\log\log n)$:

$$\begin{array}{ll} \max & \sum_{\text{rectangle } s} y_s \\ \text{s.t.} & \sum_{s \text{ contains } p} y_s \leq 1 \quad \forall \text{ point } p \\ & 0 \leq y_s \leq 1 \end{array}$$

- But the 2 LPs are dual! Q.E.D.

THE END