Computational Geometry for Non-Geometers:

Recent Developments on Some Classical Problems

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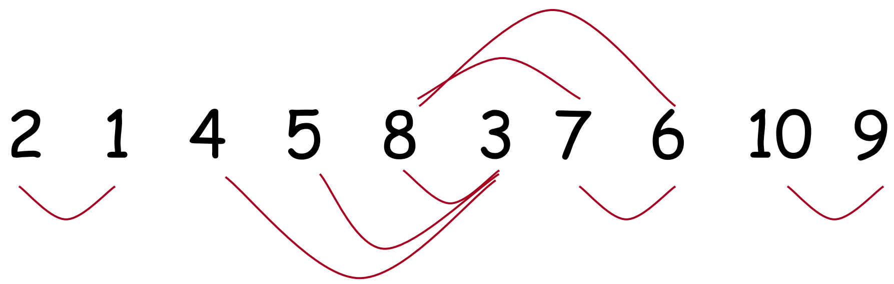
4 Toy Problems

(exercises in divide-&-conquer/data structures…)

Problem 1: Inversion Counting

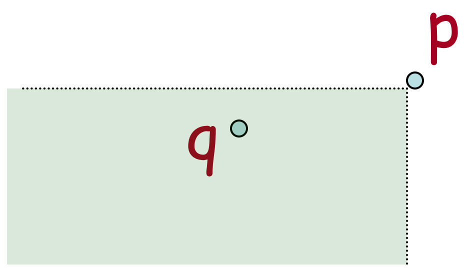
- Given permutation π of $\{1,\dots,n\}$, count # of pairs (i,j) with $i < j$ & $\pi(i) > \pi(j)$



- Exercise: $O(n \log n)$ time

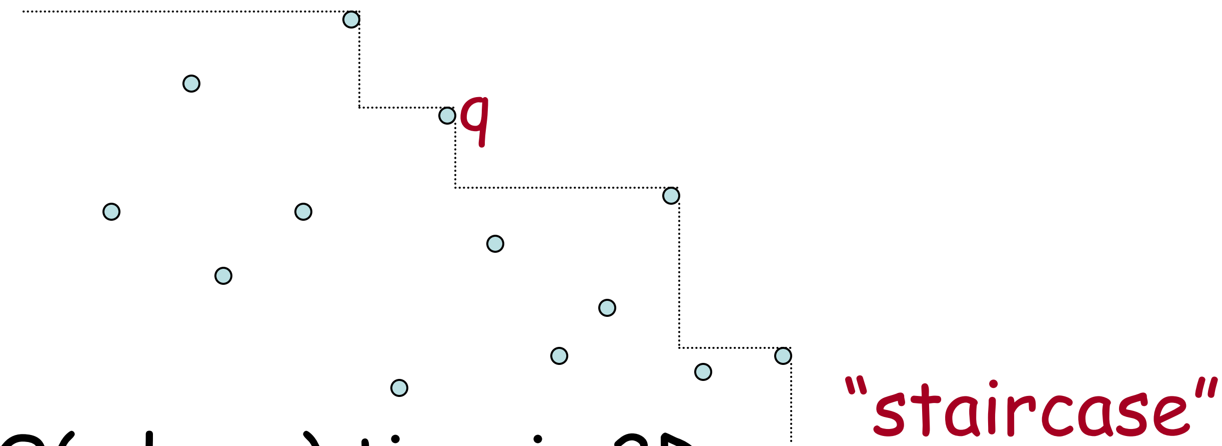
Problem 2: Maxima

- Given point set S in 2D,
  - p=(x,y) dominates q=(x',y') if x > x' & y > y'



Problem 2: Maxima

- Given point set S in 2D,
  - p=(x,y) dominates q=(x′,y′) if x > x′ & y > y′
  - q in S is maximal if no pt in S dominates q
  - find all maximal points in S

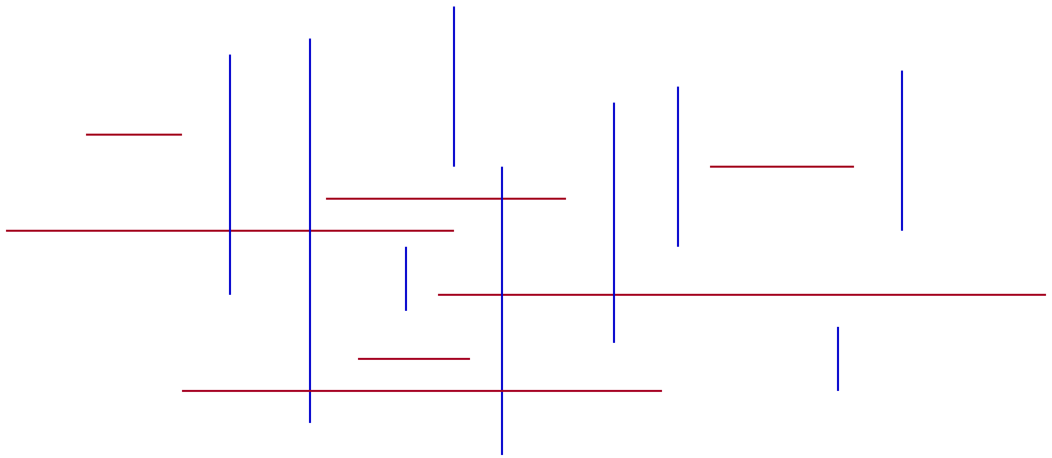


- Exercise: O(n log n) time in 2D

[History: Kung,Luccio,Preparata'75, Gabow,Bentley,Tarjan'84]

Problem 3: Orthogonal Segment Intersection

- Given n horizontal/vertical line segments in 2D, report all intersections

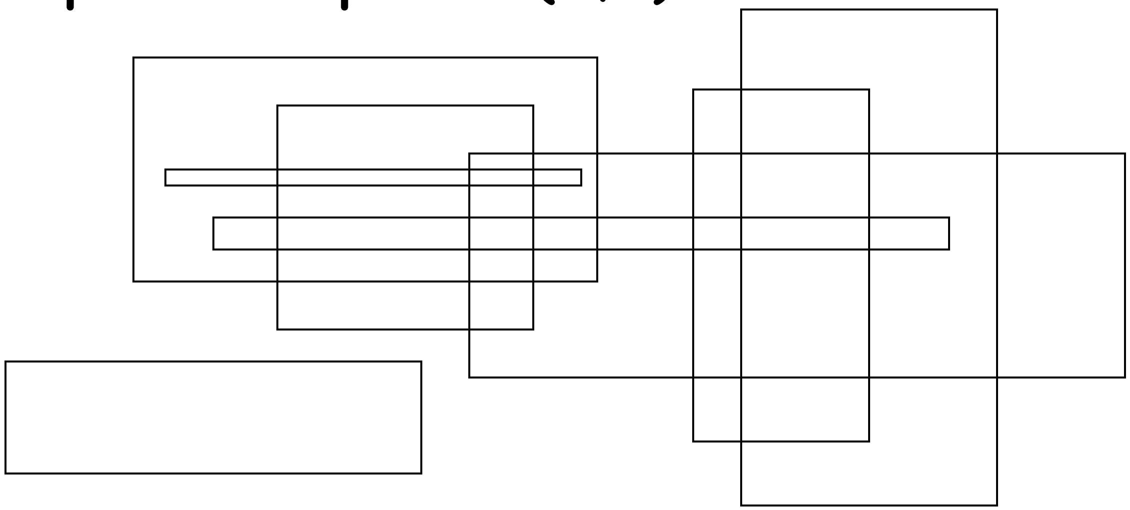


- Exercise: $O(n \log n + k)$ time (k = output size)

[History: Bentley,Ottmann'79, Overmars'87,...]

Problem 4: Rectangle Enclosure

- Given n axis-aligned rectangles in 2D, report all pairs (r,s) where r encloses s



- Exercise: O(n polylog n + k) time (k = output size)

  [History: Bentley,Wood'80, Vaishnavi,Wood'80, Lee,Preparata'82, Gupta,Janardan,Smid,DasGupta'95]

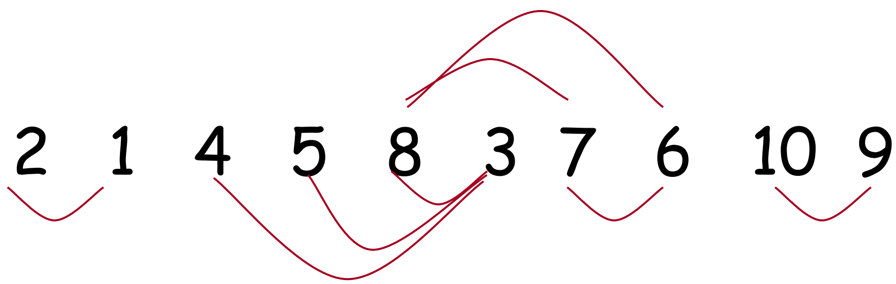
Common Thread:
Dominance Searching

- All 4 problems reduce to 2D/3D/4D (red-blue) dominance counting/existence/reporting…

  [which in turn are examples of (offline) orthogonal range searching]

Problem 1: Inversion Counting

- Given permutation π of {1,...,n}, count # of pairs (i,j) with i < j & π(i) > π(j)



point (j, -π(j)) dominates point (i, -π(i)) in 2D

- Exercise: O(n log n) time

Problem 2: Maxima

- Given point set S in 2D,
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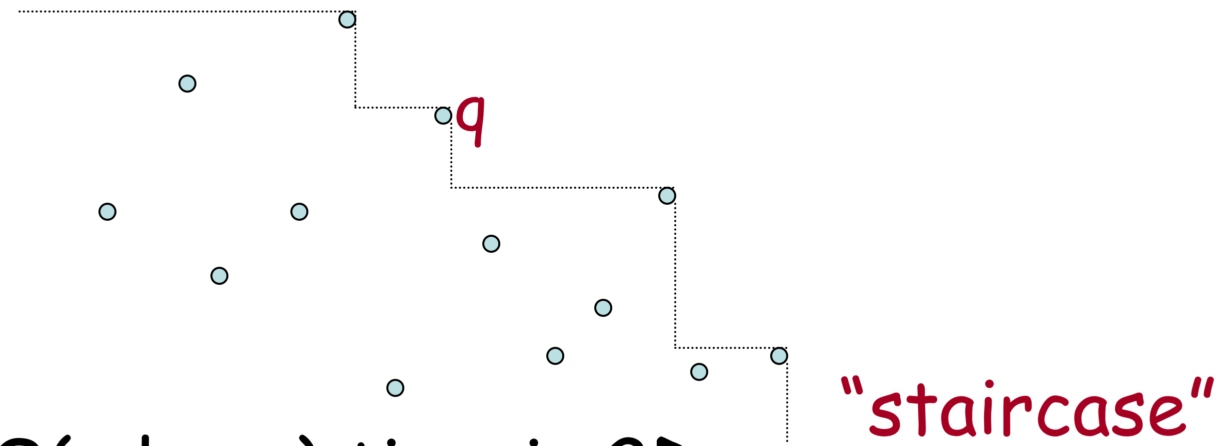

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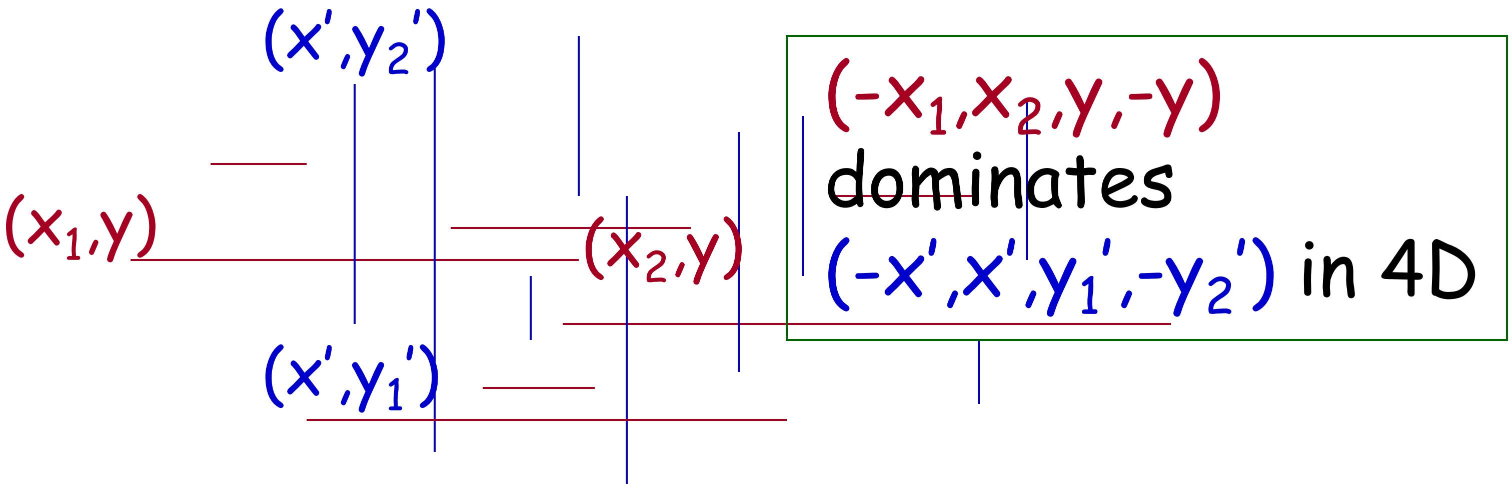

- Exercise: O(n log n + k) time (k = output size)

[History: Bentley,Ottmann'79, Overmars'87,...]

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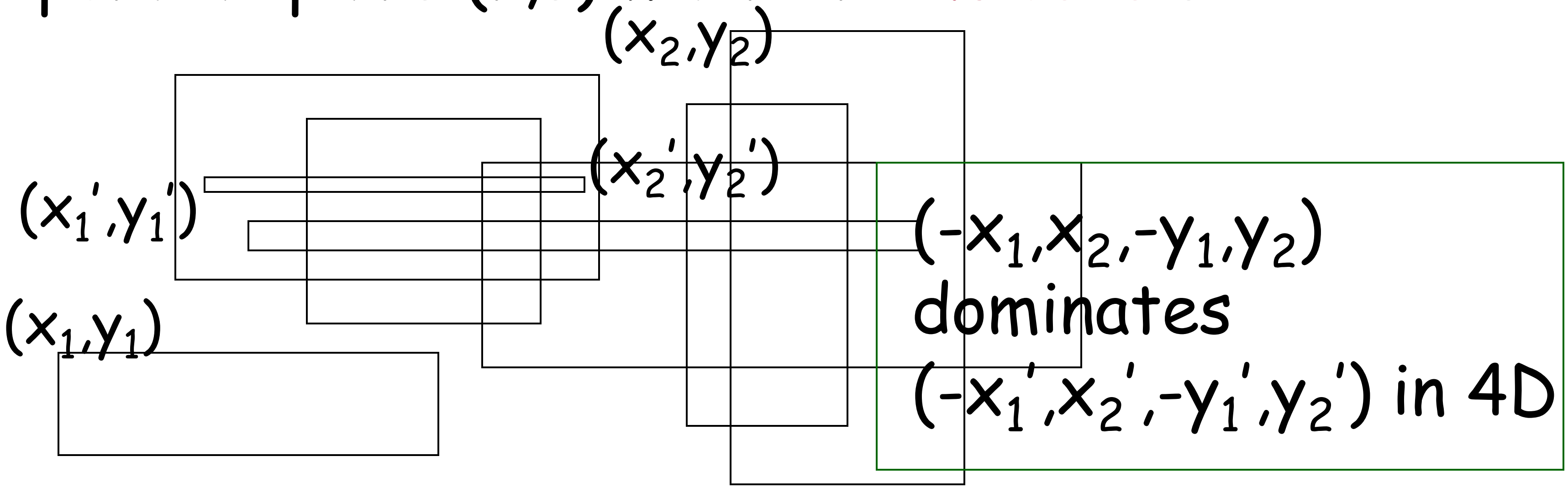

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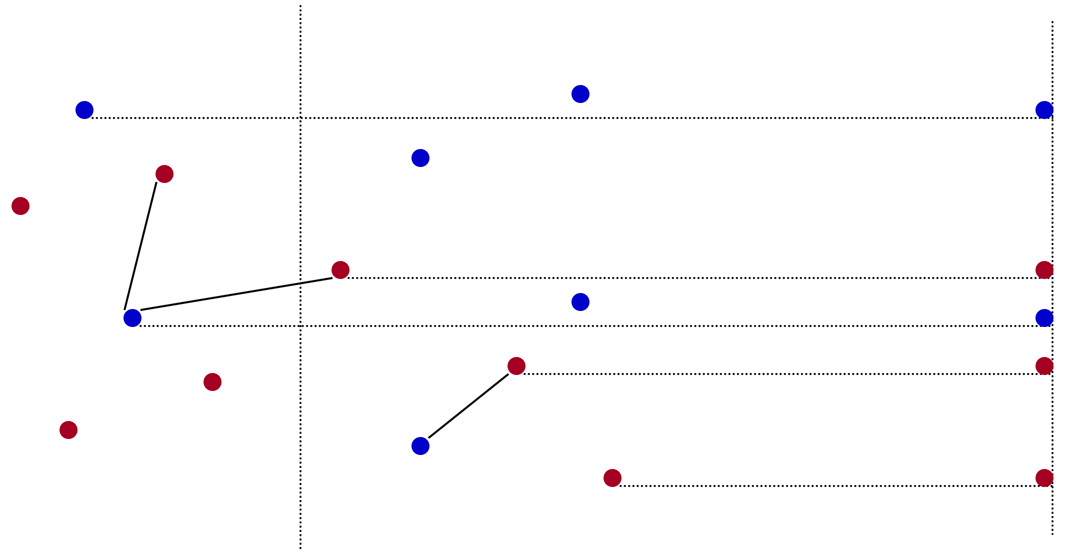
[History: Bentley,Wood'80, Vaishnavi,Wood'80, Lee,Preparata'82, Gupta,Janardan,Smid,DasGupta'95]

The "Obvious" Divide-&-Conquer Alg'm

- For (red-blue) dominance counting/existence/reporting
- In 2D:



$$T(n) = 2\,T(n/2) + O(n \log n) \Rightarrow O(n \log^2 n)$$

- By pre-sorting:

$$T(n) = 2\,T(n/2) + O(n) \Rightarrow O(n \log n)$$

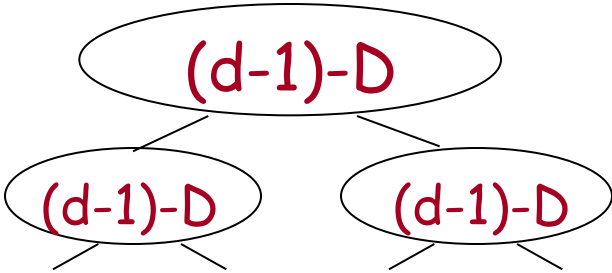
[+k for reporting]

The "Obvious" Divide-&-Conquer Alg'm

- In d-D:

  $T_d(n) = 2T_d(n/2) + T_{d-1}(n) + O(n)$

  $\Rightarrow T_d(n) = O(n \log^{d-1} n)$



- Side Remark: sol'n not tight for nonconst d...

  $T_d(n) \leq 2^{O(d)} n^{1+\varepsilon}$

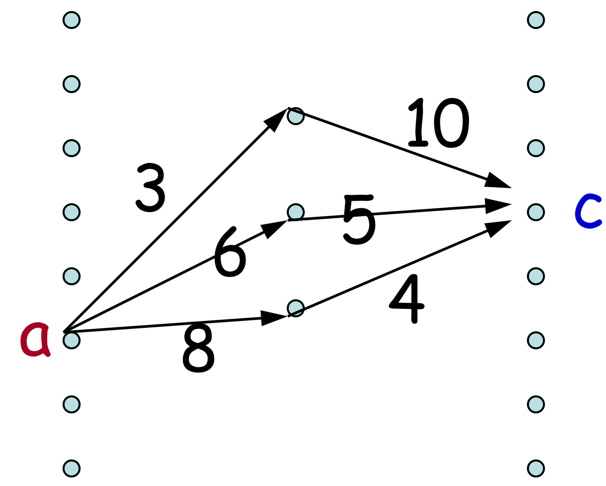
  e.g., for $d \leq \delta \log n$, $T_d(n) = O(n^{1+\varepsilon})$

  [Hint: change of var $m = c^d n$

  $T'(m) \leq 2\,T'(m/2) + T'(m/c) + O(m)$]

Detour: Appl'n to All-Pair Shortest Paths

- Given real-weighted directed tripartite graph ($A \cup B \cup C$, E), |A|=|C|=n, |B|=d, find shortest (length-2) path from every a in A to every c in C



- Trivial sol'n: $O(dn^2)$ time

- Better sol'n: Fix b_0 in $B = \{b_1,\ldots,b_d\}$.

  Find all shortest paths that use b_0, i.e., find all a in A, c in C s.t. $\forall i=1,\ldots,d$, $w(a,b_0) + w(b_0,c) < w(a,b_i) + w(b_i,c)$

i.e. $\langle w(a,b_i) - w(a,b_0)\rangle_i$ dominates $\langle w(b_0,c) - w(b_i,c)\rangle_i$ in d-D

Detour: Appl'n to APSP (Cont'd)

s.t. $\forall i=1,\ldots,d,\ \ w(a,b_0) + w(b_0,c) < w(a,b_i) + w(b_i,c)$

i.e. $\langle w(a,b_i) - w(a,b_0)\rangle_i$ dominates $\langle w(b_0,c) - w(b_i,c)\rangle_i$ in d-D

$\Rightarrow$ for $d = \delta \log n$, time $o(n^2) + k$

$\Rightarrow$ total time for all b_0: $O(n^2)$

$\Rightarrow$ general APSP in $O(n^3 / \log n)$ time [C.,WADS'05]

[improved over … Floyd-Warshall'59, Friedman'76, Takaoka'92, Dobosiewicz'90, Han'04, Takaoka'04,'05, Zwick'04]

- Current record: $O(n^3 \log^3 \log n / \log^2 n)$ [C.,STOC'07]

…Back to Dominance in Low-D…

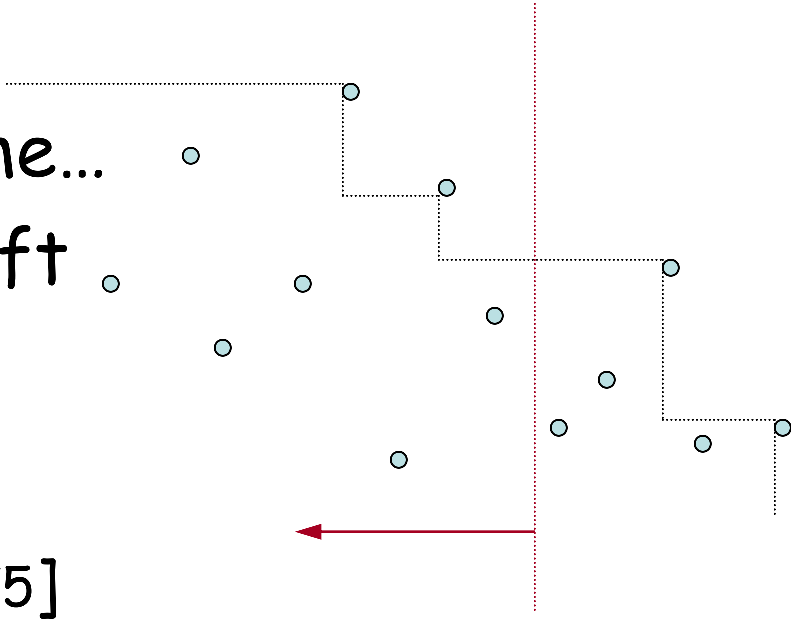
- How to beat $O(n \log^{d-1} n)$?
- Rest of Talk:

  I. Faster Worst-Case Alg'ms

  II. Beyond Worst-Case Alg'ms ("Instance Optimality")

Improvement 1 (Maxima/Dominance Existence)

- Assume pre-sorted input
- Solve 2D problem in O(n) time... by sweeping from right to left

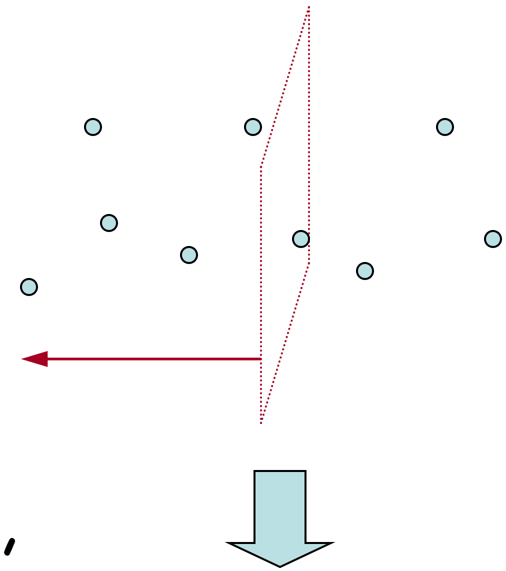


$\Rightarrow$ in d-D, $O(n \log^{d-2} n)$ time

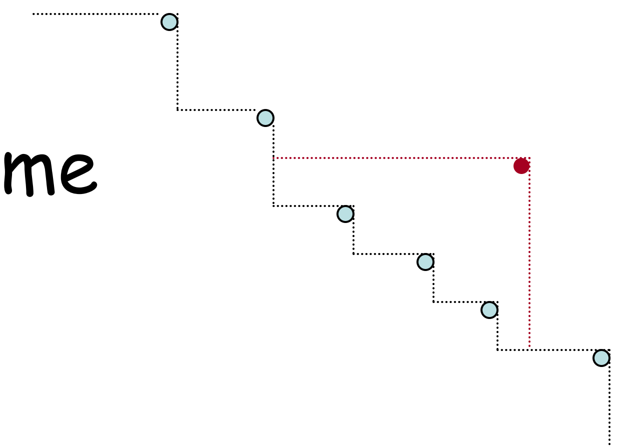
[Kung,Luccio,Preparata,JACM'75]

Improvement 2 (Maxima/Dominance Existence)

- Solve 3D problem faster by sweeping from right to left… & using data structures…



with van Emde Boas (vEB) trees, in O(n loglog n) time



⇒ in d-D, $O(n \log^{d-3} n \log\log n)$ time [Gabow,Bentley,Tarjan,STOC'84]

Better? No progress since…

A New Result [C.,Larsen,Pătraşcu'11]

- 4D problem can be solved in $O(n \log n)$ time

$\Rightarrow$ in d-D, $O(n \log^{d-3} n)$ time

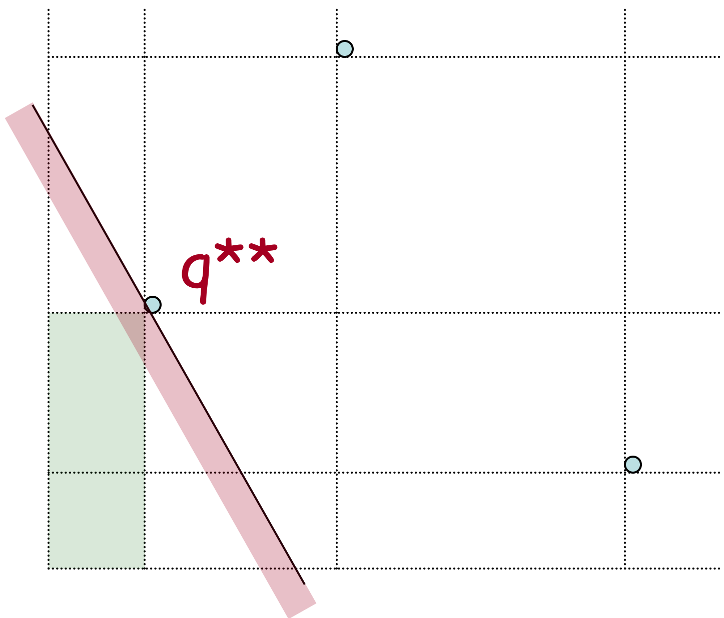
Theorem [Chazelle,FOCS'89]

- Given 2 convex polyhedra in 3D with n vertices, we can compute their intersection, or their convex hull, in O(n) time

[Proof: 26-page paper!]

Connection of Maxima to Convex Hull

- Assume coords are in {1,...,n} (by pre-sorting)
- Map $p=(i,j)$ to $p^* = (3^i, 3^j)$
- Map $q=(a,b)$ to halfspace
  $q^{**} = \{ x/3^a + y/3^b \leq 2 \}$



- p is dominated by q
  $\Leftrightarrow$ $i \leq a$ & $j \leq b$ $\Leftrightarrow$ p^* lies in q^{**}
- minimal points of P
  $\Leftrightarrow$ vertices of (lower-left) convex hull of P*
- dominance range searching in P
  $\Rightarrow$ halfspace range searching in P*

Aside: History of DSs for 3D Halfspace/Dominance Range Reporting

	space	query time
...		
Aggarwal,Hansen,Leighton [STOC'90]	n log n	log n + k
C. [FOCS'98], Ramos [SoCG'99]	n loglog n	log n + k
Makris,Tsakalidis'98	n	log n loglog n + k
Nekrich [SoCG'07]	n log n	$\log^3 \log n + k$
Afshani [ESA'08]	n	$\log^2 \log n + k$
Afshani,Chan [SODA'09]	n	log n + k
C. [SODA'11]	n	loglog n + k

A New 4D Dominance Existence Alg'm

- Divide-&-conquer ⇒ $T_4(n) = 2T_4(n/2) + T_3(n) + O(n)$
- To solve 3D red-blue dominance existence subproblem:

  not ∃ pair (p,q): p is dominated by q

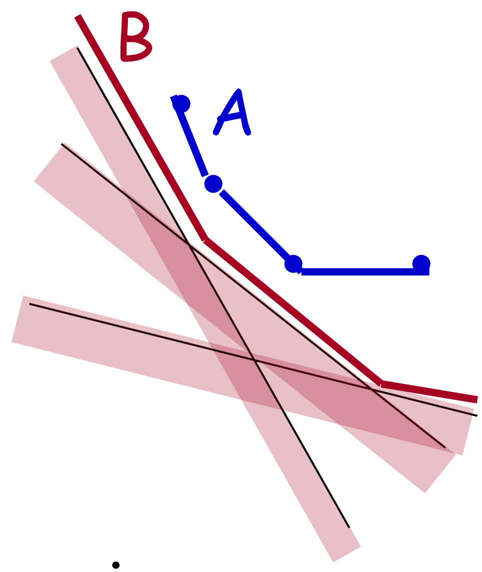
  ⇔ not ∃ p,q: point p* lies in halfspace q**

  ⇔ (convex hull A of) all p* lie in (intersection B of) complement of all q**

  ⇔ $A \cap B = A$

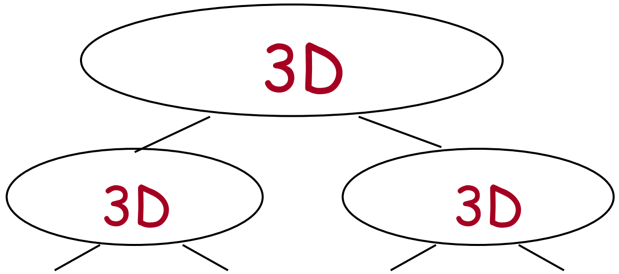
  O(n) time by Chazelle's alg'm!

  ... provided that convex polyhedra A & B are given



4D Alg'm (Cont'd)

- but can pre-compute all A's & B's
  by bottom-up merging...
  by Chazelle's alg'm again!



- pre-computation: $T(n) = 2T(n/2) + O(n) \Rightarrow O(n \log n)$
- rest: $T(n) = 2T(n/2) + O(n) \Rightarrow$ $O(n \log n)$ YES!

- Rmk: more complicated alg'ms [C.,Larsen,Pătraşcu'11] to get
  $O(n \log n)$ for 4D maxima (Problem 2) &
  $O(n \log n + k)$ for 2D rectangle enclosure (Problem 4)

  [need not Chazelle's alg'm, but Clarkson-Shor random sampling, shallow cuttings, higher-deg. range trees, + bit packing tricks, ... (yikes!)]

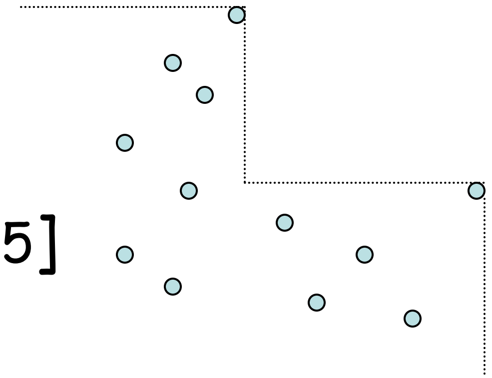
- Rest of Talk:

    I. Faster Worst-Case Alg'ms

    II. Beyond Worst-Case Alg'ms

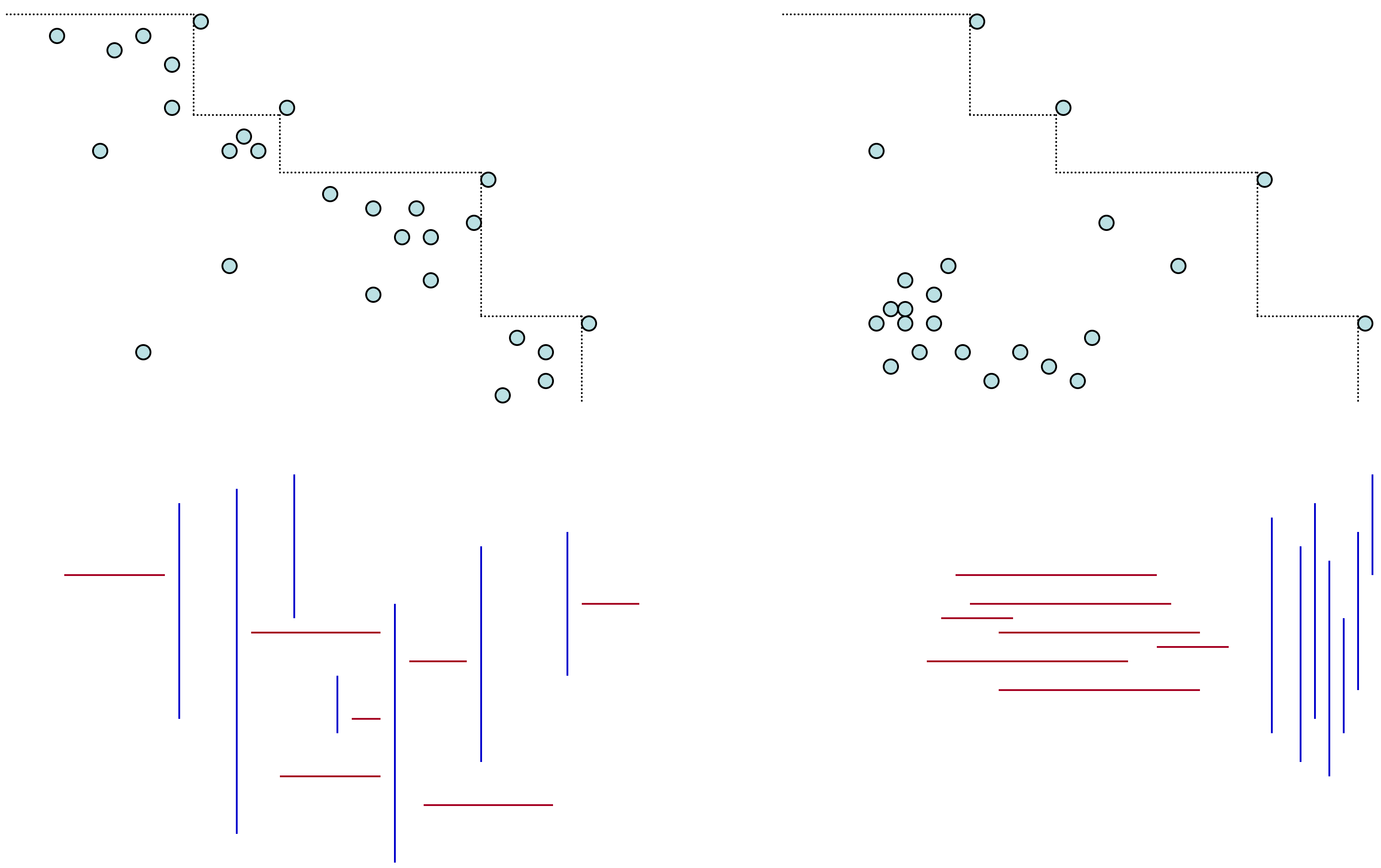
The 2D Maxima Problem, Revisited

- $\Omega(n \log n)$ worst-case lower bound under comparison/decision tree model, but...
- Output-sensitive alg'ms:
  - $O(nh)$ is easy, where h = output size
  - $\Theta(n \log h)$ [Kirkpatrick,Seidel,SoCG'85]



- "Average-case" alg'ms:
  - for uniformly distributed pts in a square, $O(n)$ expected [Bentley,Clarkson,Levine,SODA'90; Golin'94; Clarkson,FOCS'94; ...]

“Easy” vs. “Hard” Input



New Result [Afshani,Barbay,C.,FOCS'09]

- ∃ alg'm for the 2D maxima problem that beats all other alg'ms on all point sets simultaneously !

an "instance-optimal" alg'm

Def'n of Instance Optimality (1st Attempt)

- Let $T_A(S)$ = time of alg'm A on input sequence S
- Let $OPT(S)$ = min $T_A(S)$ over all alg'ms A
- A is instance-optimal if $\forall S$, $T_A(S) \leq O(1) \cdot OPT(S)$

… but not possible for 2D maxima !
[for every input sequence S, there is an alg'm with runtime O(n) on S]

Our Def'n of "Instance Optimality"

- Let $T_A(S)$ = ~~max~~ average time of alg'm A over all permutations of input set S

- Let $OPT(S)$ = min $T_A(S)$ over all alg'ms A

- A is instance-optimal in the ~~order-oblivious~~ random-order setting if $\forall S,\ T_A(S) \le O(1) \cdot OPT(S)$

  [subsumes output-sensitive alg'ms, & all alg'ms that do not exploit input order, ...]

  [& average-case alg'ms for all point distributions as well!]

Related Work on Instance Optimality

- Fagin,Lotem,Naor'03 [finding the top k elements under a monotone aggregate scoring function]
- Sleator,Tarjan'85's "dynamic optimality conjecture" for binary search trees
- Competitive analysis of on-line alg'ms
- Various adaptive alg'ms, e.g.,

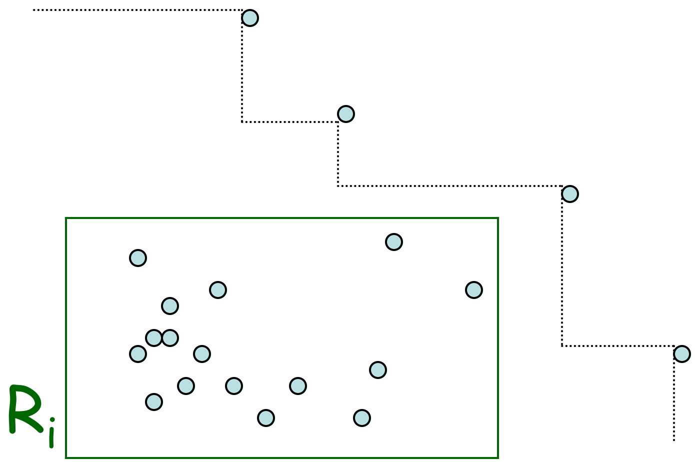
  Demaine,Lopez-Ortiz,Munro'00 [set union/intersection], Baran,Demaine'04 [approx. distance from pt to black-box curve], ...

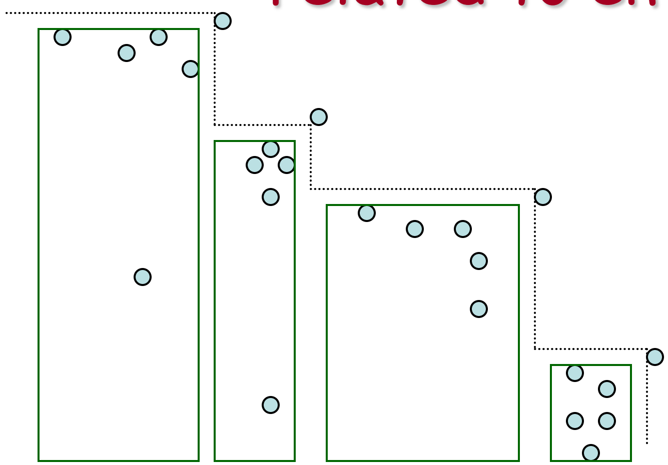
A Measure of Difficulty

- Given point set S of size n
- Consider a partition P of S into subsets S_i s.t.

  each subset S_i can be enclosed in a rectangle R_i that is below staircase(S) (*)

- Let $H(P) := \sum_i |S_i| \log(n/|S_i|)$ ← related to entropy





$H(P) \sim n + (h \log n)$

$H(P) \sim h \cdot (n/h) \log h = n \log h$

A Measure of Difficulty

- Given point set S of size n
- Consider a partition P of S into subsets S_i s.t.

  each subset S_i can be enclosed in (*)
  a rectangle R_i that is below staircase(S)

- Let $H(P) := \sum_i |S_i| \log(n/|S_i|)$

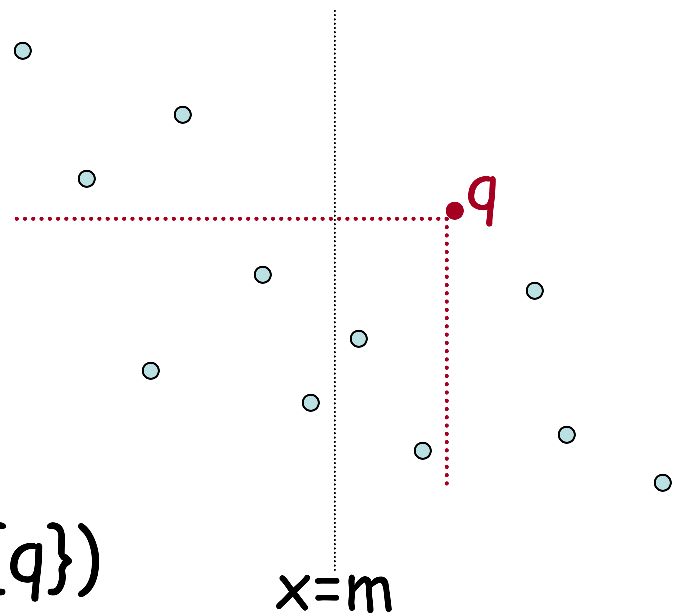
- Define the difficulty of S to be

  $H(S) :=$ min $H(P)$ over all valid partitions P satisfying (*)

An Instance-Optimal 2D Maxima Alg'm

Maxima(S):

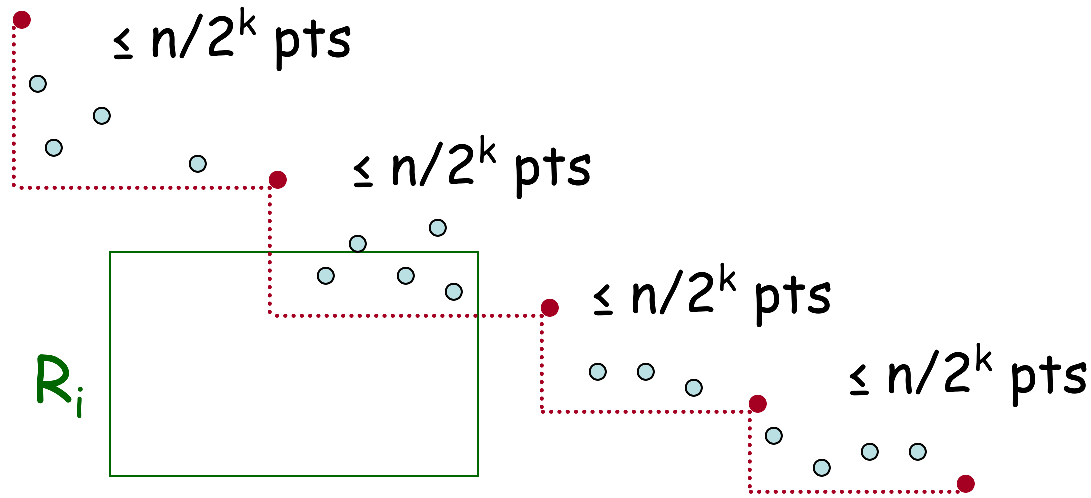
1. if $|S| \leq 2$ then return ...
2. m = x-median
3. q = highest pt right of x=m
4. prune all pts dominated by q
5. Maxima({all pts left of q} $\cup$ {q})
6. Maxima({all pts right of q} $\cup$ {q})



- Rmk: this is not new — same as Kirkpatrick,Seidel'85's output-sensitive alg'm !!

Analysis

- At level k of recursion:



- Let P be any valid partition
- Let S_i be any subset of P, enclosed in rectangle R_i

$\Rightarrow$ # pts in S_i that survive level k $\leq \min\{n/2^k, |S_i|\}$

$\Rightarrow$ total # pts that survive level k $\leq O(\sum_i \min\{n/2^k, |S_i|\})$

Analysis (Cont'd)

$\Rightarrow$ total # pts that survive level k $\leq O(\sum_i \min\{n/2^k, |S_i|\})$

$\Rightarrow$ runtime $\leq O(\sum_k \sum_i \min\{n/2^k, |S_i|\})$

$= O(\sum_i \sum_k \min\{n/2^k, |S_i|\})$

$= O(\sum_i (|S_i| + \ldots + |S_i| + |S_i|/2 + |S_i|/4 + \ldots))$

log $(n/|S_i|)$ times

$= O(\sum_i |S_i| \log (n/|S_i|)) = O(H(P))$

$\Rightarrow$ runtime $\leq O(\min_P H(P)) = O(H(S))$ GOOD!

Lower Bound

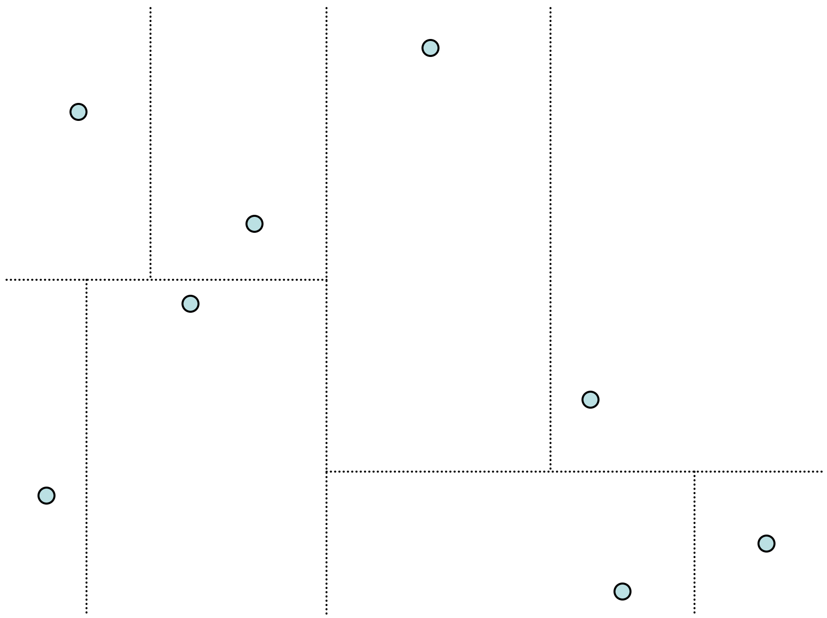
- Standard $\Omega(n \log n)$ proofs can't show instance-specific lower bounds…
- 2 $\Omega(H(S))$ Proofs [Afshani,Barbay,C.'09]
  - An encoding-based argument
  - An adversary-based argument

Lower Bound Proof

- Build k-d tree

  cell at depth k
  contains $n/2^k$ pts

  i.e., depth of cell R
  $= \log (n/|S \cap R|)$



- Make cell R a leaf if R is below staircase(S)
  $\Rightarrow$ leaf cells yield a valid partition P*
- Adversary simulates alg'm on unknown input
- Maintain a cell R_p for each input pt p (initially, R_p = root)

Lower Bound Proof (Cont'd)

- When alg'm makes, say, x-comp. betw'n p & q:

  if depth(R_p) is odd then $R_p \leftarrow$ any child of R_p

  if depth(R_q) is odd then $R_q \leftarrow$ any child of R_q

  if x-median(R_p) < x-median(R_q) then

  $R_p \leftarrow$ left child of R_p & $R_q \leftarrow$ right child of R_q

  declare " < "

  else symmetric

- When R_p becomes a leaf, fix p to an unassigned pt in $S \cap R_p$

  [Note: don't let more than $|S \cap R|$ points go into cell R...]

$\Rightarrow$ At the end, get a permutation of S

Lower Bound Proof (Cont'd)

- Let $D = \sum_{p \text{ in } S} \text{depth}(R_p)$
- Each comp. increases D by O(1) $\Rightarrow$ $D \leq O(\#\text{ comps})$
- At the end, each R_p must be a leaf
  (otherwise staircase could change)

$\Rightarrow$ # comps $\geq \Omega(D) = \Omega(\sum_{\text{leaf } R} |S \cap R| \text{ depth}(R))$

$= \Omega(\sum_{\text{leaf } R} |S \cap R| \log (n/|S \cap R|))$

$= \Omega(H(P^*)) \geq \Omega(H(S))$ Q.E.D.!

Other Instance-Optimal Results

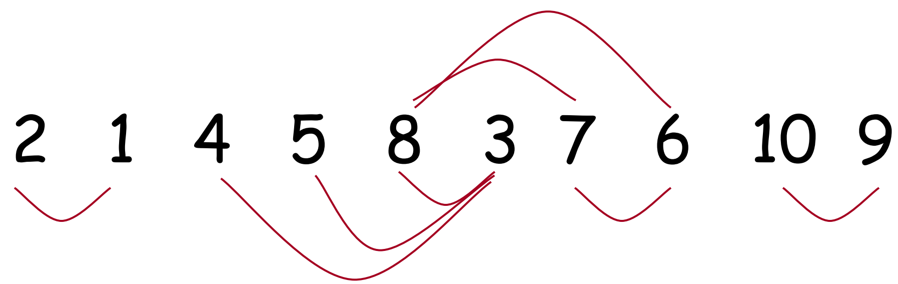
- 3D/4D maxima: need a new alg'm this time, explicitly using k-d trees...
- 2D orthogonal segment intersection
- 2D red-blue rectangle enclosure
- & classical non-orthogonal problems too !
  [2D/3D convex hull, 2D point location, 2D/3D halfspace range reporting, ... under a multilinear decision tree model]

Conclusions

- find more instance-optimal results?
- worst-case complexity of Problems 1 & 3??

Problem 1: Inversion Counting

- Given permutation π of $\{1,\dots,n\}$,
  count # of pairs (i,j) with $i < j$ & $\pi(i) > \pi(j)$

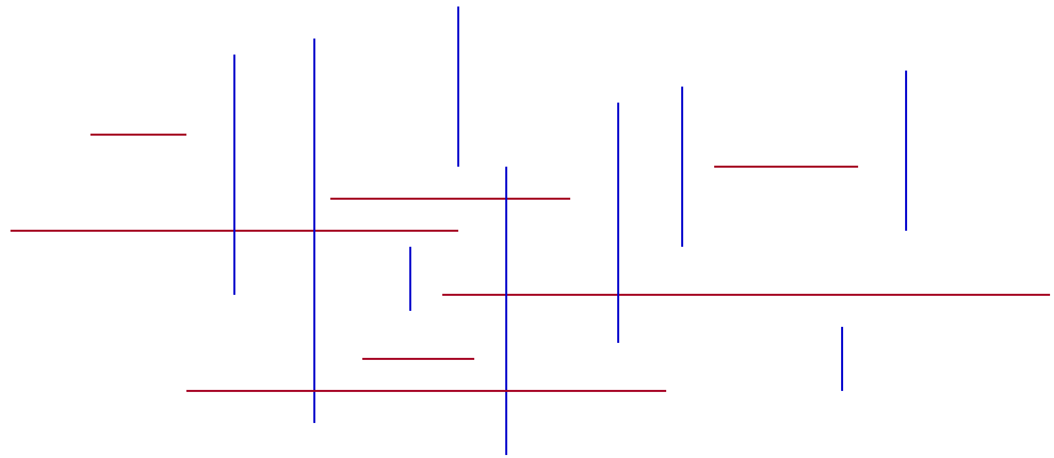


- Exercise: $O(n \log n)$ time

Current record: $O(n \log^{1/2} n)$ time [C.,Pătraşcu,SODA'10]
Can you do better??

Problem 3: Orthogonal Segment Intersection

- Given n horizontal/vertical line segments in 2D, report all intersections



- Exercise: $O(n \log n + k)$ time (k = output size)
[History: Bentley,Ottmann'79, Overmars'87,...]

For pre-sorted input, $O(n \log\log n + k)$ by vEB trees
But is $O(n + k)$ time possible??

The End