3SUM and Related Problems

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The 3SUM Problem

- Given sets A, B, C of n real numbers, decide $\exists a \in A$, $b \in B$, $c \in C$ with

$$a + b = c$$

- Standard Sol'n 1:

  sort $A + B$, sort C, merge

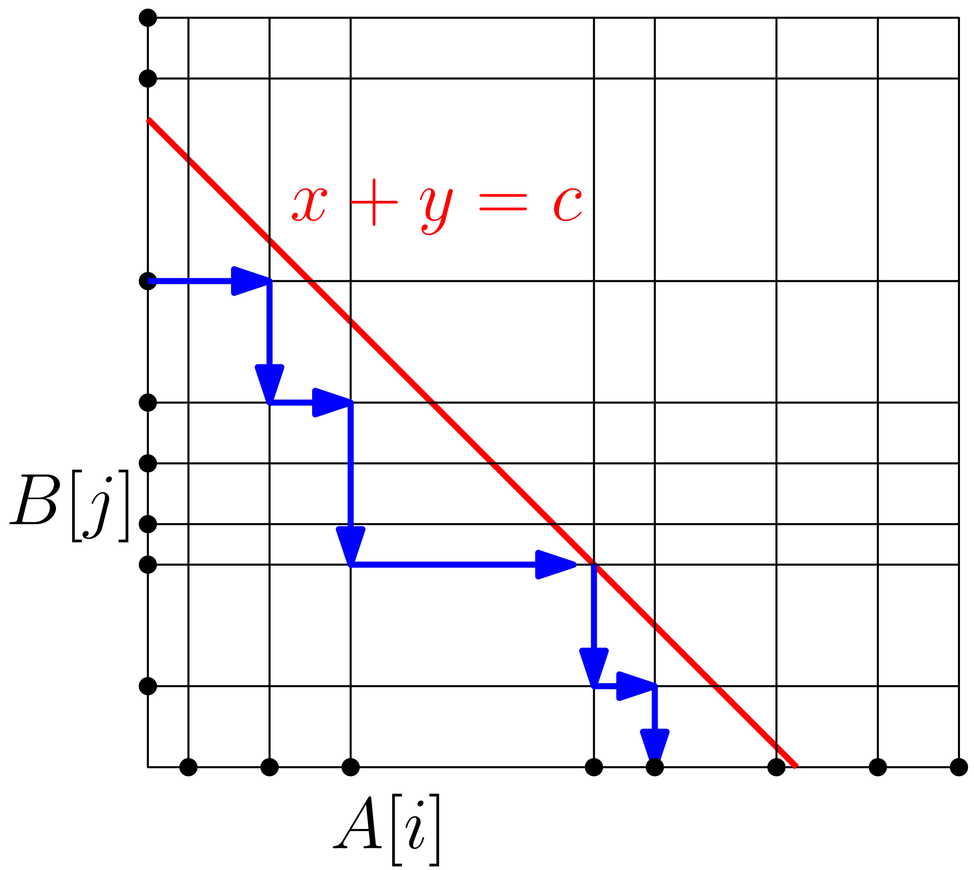
  $\Rightarrow \boxed{O(n^2 \log n)}$ time

The 3SUM Problem

- Standard Sol'n 2:

  Preprocessing: sort A & B

  For each $c \in C$: test if $c \in A + B$ in $O(n)$ time



The 3SUM Problem

- Standard Sol'n 2:

  Preprocessing: sort A & B

  For each $c \in C$: test if $c \in A + B$ in $O(n)$ time

  $\Rightarrow$ $\boxed{O(n^2)}$ time (better??)

Variants

- monochromatic version: $A = B = C$
- integer version
- convolution 3SUM: given sequences A, B, C, decide $\exists i, k$ s.t. $A[i] + B[k - i] = C[k]$
- kSUM
  ($O(n^{\lceil k/2 \rceil})$ time for k odd, $O(n^{k/2} \log n)$ time for k even)

Lower Bound Conjecture

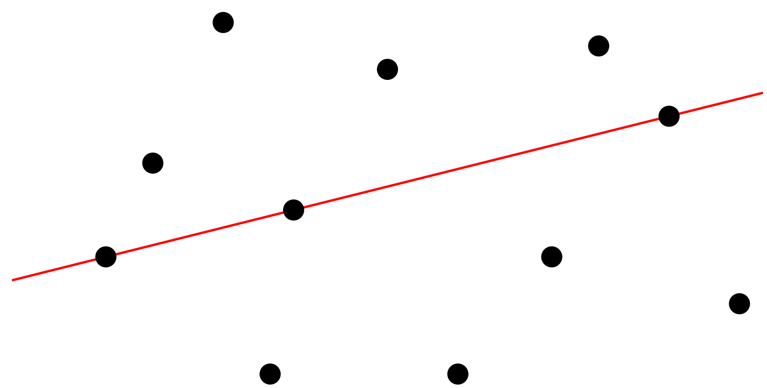
- 3SUM requires $\Omega(n^{2-\varepsilon})$ time??

  (or more strongly, 3SUM for integers (in $[n^2]$) requires $\Omega(n^{2-\varepsilon})$ time??)

- useful for proving conditional polynomial lower bound for other problems, by reductions from 3SUM...

Exs of 3SUM-Hard Problems (started by Gajentaan–Overmars'93)

- 3COLLINEAR: given n points in 2D, decide $\exists$ 3 collinear points



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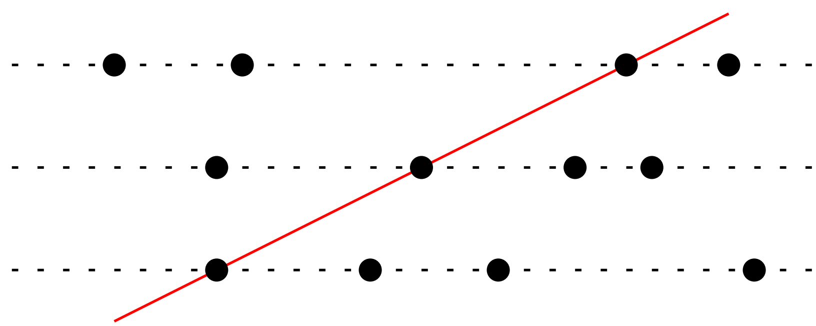


- 3CONCURRENT: given n lines in 2D, decide $\exists$ 3 concurrent lines

Exs of 3SUM-Hard Problems (started by Gajentaan–Overmars'95)

- given n points in 2D, find 3 points defining smallest triangle area
- given n triangles in 2D, decide if union covers $[0, 1]^2$
- given n halfplanes in 2D, find the deepest point
- given n red/blue points in 2D, find smallest # pts to remove s.t. $\exists$ line separating red from blue points
- given n line segment obstacles in 2D and initial/final position of a rod, decide if motion exists for the rod
- given n points in 3D, find the max Tukey depth
- given 3 polygons in 2D, decide if their common intersection is empty

$\vdots$

Exs of Integer-3SUM-Hard Problems

- zero-weight triangle in an edge-weighted graph: $O(n^{3-\delta})$ time? [Vassilevska–Williams (FOCS'10)]
- enumerate t triangles in a graph: $O(t^{1/3}m^{1-\delta})$ time? [Pătraşcu (STOC'10); Kopelowitz–Pettie–Porat (SODA'16)]
- dynamic reachability & dynamic subgraph connectivity: $n^{o(1)}$ query & update time? [Pătraşcu (STOC'10)]
- dynamic max matching: similar [Abboud–Vassilevska (FOCS'14); Kopelowitz–Pettie–Porat (SODA'16)]
- local alignment: $O(n^{2-\delta})$ time? [Abboud–Vassilevska–Weimann (ICALP'14)]
- jumbled string indexing over $[\sigma]$: n queries in $O(n^{2-1/(\sigma-1-\delta)})$ time? [Amir–C.–Lewenstein–Lewenstein (ICALP'14)]

$\vdots$

(all via Integer Convolution 3SUM [Pătraşcu (STOC'10)]. . .)

Lower Bound Results

- Erickson [SODA'95]: 3SUM needs $\boxed{\Omega(n^2)}$ comparisons for 3-linear decision trees

(kSUM needs $\Omega(n^{\lceil k/2 \rceil})$ comps for k-linear decision trees)

(Ailon–Chazelle [STOC'04]: kSUM needs $n^{\Omega(k)}$ for $(k + O(1))$-linear decision trees)

(~~Erickson–Seidel [FOCS'93]:~~ Erickson [SoCG'96]: 3COLLINEAR needs $\Omega(n^2)$ sidedness tests)

The Surprise…

- Grönlund–Pettie [FOCS'14]:

  3SUM can be solved in $\boxed{\widetilde{O}(n^{3/2})}$ comparisons for 4-linear decision trees

  & has an $\boxed{O(n^2/\log^{\Omega(1)} n)}$ time alg'm…

Subsequent Results

- Decision trees

Grönlund–Pettie [FOCS'14]	$O(n^{3/2}\sqrt{\log n})$
Gold–Sharir'15	$O(n^{3/2})$
Kane–Lovett–Moran [STOC'18]	$O(n \log^2 n)$

- Alg'ms

Grönlund–Pettie [FOCS'14]	$O^*(n^2/\log^{2/3} n)$ det.
	$O^*(n^2/\log n)$ rand.
Freund'15/Gold–Sharir'15	$O^*(n^2/\log n)$ det.
C. [SODA'18]	$\boxed{O^*(n^2/\log^2 n)}$ det.

Rest of Talk

1. Subquadratic Decision-Tree Upper Bounds

2. Slightly Subquadratic Alg'ms

3. Extensions to Other Problems

Fredman's Trick

$$a + b \ \leq \ a' + b'$$

$$\Updownarrow$$

$$a - a' \ \leq \ b' - b$$

(Fredman [FOCS'75] used this to prove $\widetilde{O}(n^{5/2})$ decision tree upper bound for all-pairs shortest paths & $(\min, +)$-matrix multiplication)

Warm-Up: (min, +)-Convolution [Bremner–C.–Demaine–Erickson–Hurtado–Iacono–Langerman–Pătraşcu–Taslakian'06]

- Problem: Given sequences A, B of n real numbers, compute $C[k] = \min_i (A[i] + B[k-i])$ for all k

- Preprocessing:

  divide A into groups $A_1, \ldots, A_{n/d}$ of size d
  divide B into groups $B_1, \ldots, B_{n/d}$ of size d

  sort $\bigcup_i (A_i - A_i) \;\cup\; \bigcup_j (B_j - B_j)$

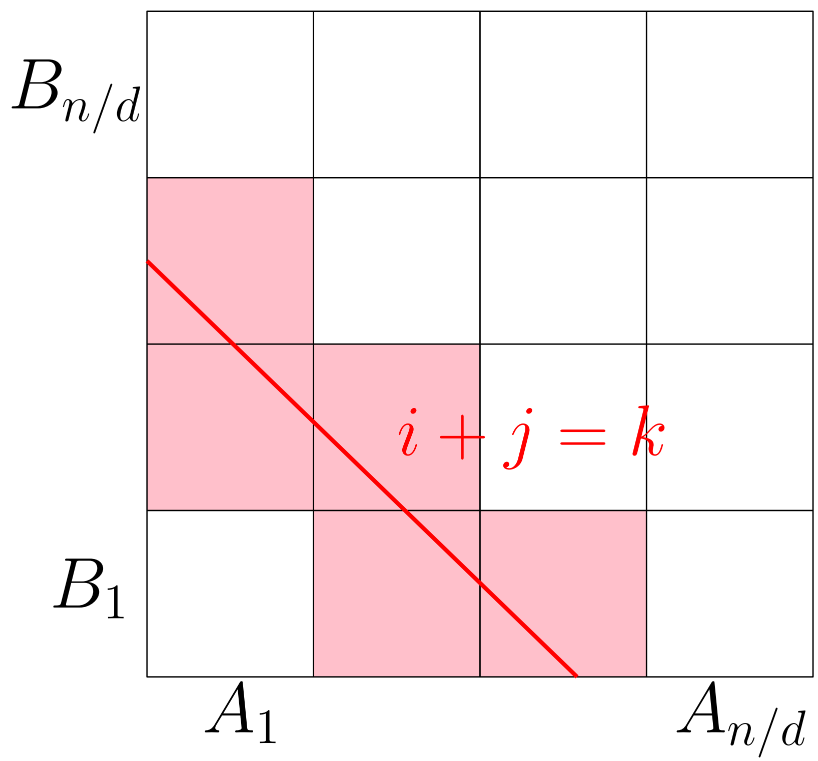
  $\Rightarrow O(n/d \cdot d^2 \cdot \log n) = \widetilde{O}(dn)$ comps

  (by Fredman's trick, comparisons internal to each $A_i + B_j$ are now free. . .)

Warm-Up: $(\min, +)$-Convolution [Bremner–C.–Demaine–Erickson–Hurtado–Iacono–Langerman–Pătraşcu–Taslakian'06]

- To compute each $C[k]$:

  find min of $O(n/d)$ elements, in $O(n/d)$ comps



Warm-Up: $(\min, +)$-Convolution [Bremner–C.–Demaine–Erickson–Hurtado–Iacono–Langerman–Pătraşcu–Taslakian'06]

- To compute each $C[k]$:

  find min of $O(n/d)$ elements, in $O(n/d)$ comps

- total # comps $\widetilde{O}(dn + n \cdot n/d)$

- set $d = \sqrt{n} \Rightarrow \boxed{\widetilde{O}(n^{3/2})}$ comps

Similar: (median, +)-Convolution [Bremner–C.–Demaine–Erickson–Hurtado–Iacono–Langerman–Pătraşcu–Taslakian'06]

- Problem: Given sequences A, B of n real numbers, compute $C[k] =$ median of $\{A[i] + B[k-i]\}_{i \in [k]}$

Similar: (median, +)-Convolution [Bremner–C.–Demaine–Erickson–Hurtado–Iacono–Langerman–Pătraşcu–Taslakian'06]

- To compute each $C[k]$:

  find median in union of $O(n/d)$ sorted lists of size d in $O((n/d)\log d)$ comps [Frederickson–Johnson'80s]

- total # comps $\widetilde{O}(dn + n \cdot n/d)$

- set $d = \sqrt{n} \Rightarrow \boxed{\widetilde{O}(n^{3/2})}$ comps

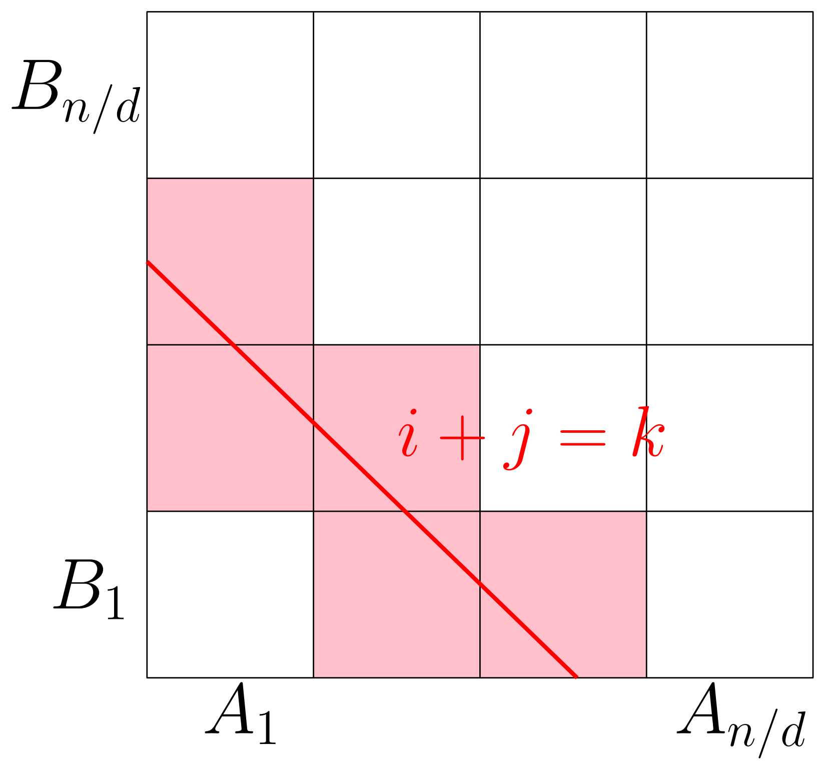
Convolution 3SUM [Grønlund–Pettie'14]

- Problem: Given sequences A, B, C of n real numbers, decide $\exists i, k$ with $C[k] = A[i] + B[k - i]$

Convolution 3SUM [Grønlund–Pettie'14]

- To search for each $C[k]$:

  search in $O(n/d)$ sorted lists of size d
  in $O((n/d)\log d)$ comps (by binary searches)



Convolution 3SUM [Grønlund–Pettie'14]

- To search for each $C[k]$:

  search in $O(n/d)$ sorted lists of size d
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- total # comps $\widetilde{O}(dn \; + \; n \cdot n/d)$

- set $d = \sqrt{n} \; \Rightarrow \; \boxed{\widetilde{O}(n^{3/2})}$ comps

Finally: 3SUM [Grønlund–Pettie'14]

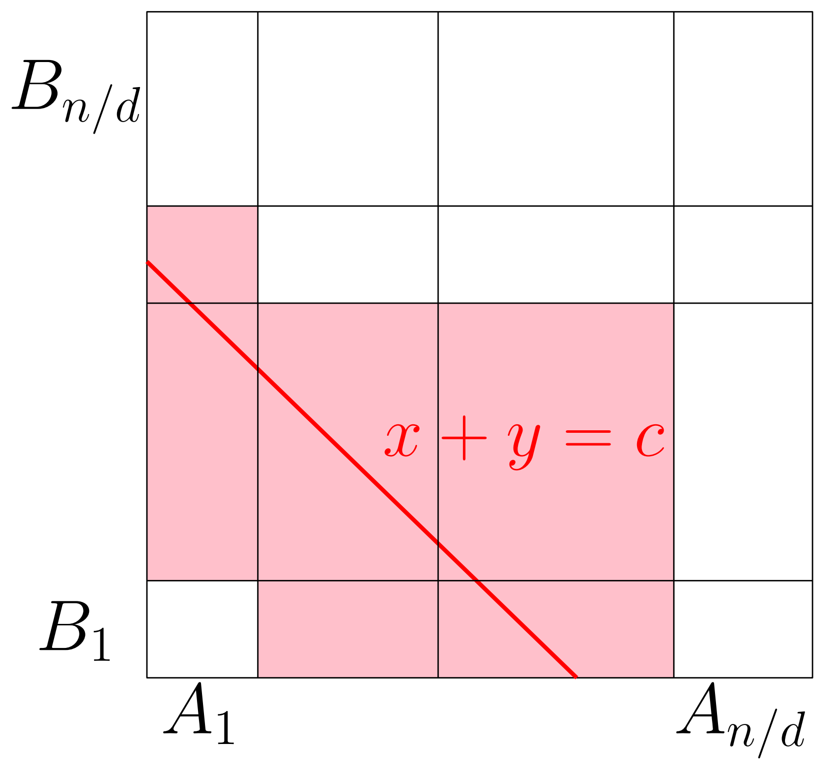
- first sort A & B
- rest is basically the same!

Finally: 3SUM [Grønlund–Pettie'14]

- To search for each $c \in C$:

  search in $O(n/d)$ sorted lists of size d^2

  in $O((n/d) \log d)$ comps (by binary searches)



Finally: 3SUM [Grønlund–Pettie'14]

- To search for each $c \in C$:

  search in $O(n/d)$ sorted lists of size d^2
  in $O((n/d)\log d)$ comps (by binary searches)

- total # comps $= \widetilde{O}(dn + n \cdot n/d)$

- set $d = \sqrt{n} \Rightarrow \boxed{\widetilde{O}(n^{3/2})}$ comps

Rest of Talk

1. Subquadratic Decision-Tree Upper Bounds

2. Slightly Subquadratic Alg'ms

3. Extensions to Other Problems

3SUM Alg'm 0

- build decision tree for small input size m
- divide A, B, C into groups of size m
  $\Rightarrow O(n/m)^2$ subproblems of size $O(m)$
- can solve each subproblem in $\widetilde{O}(m^{3/2})$ time
- total time $\widetilde{O}((n/m)^2 \cdot m^{3/2} + 2^{\widetilde{O}(m^{3/2})})$
- set $m \approx \log^{2/3} n \quad \Rightarrow \quad \boxed{O^*(n^2/\log^{1/3} n)}$ time

3SUM Alg'm 1 [C.'18]

- Key Idea: think geometrically... in d dimensions!
  (inspired by C. [STOC'07] on $(\min, +)$-matrix multiplication)

- map each group A_i to a point $(A_i[1], \ldots, A_i[d])$

- map each group B_j to $O(d^4)$ hyperplanes

  $$\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_u + B_j[v] = x_{u'} + B_j[v']\}$$

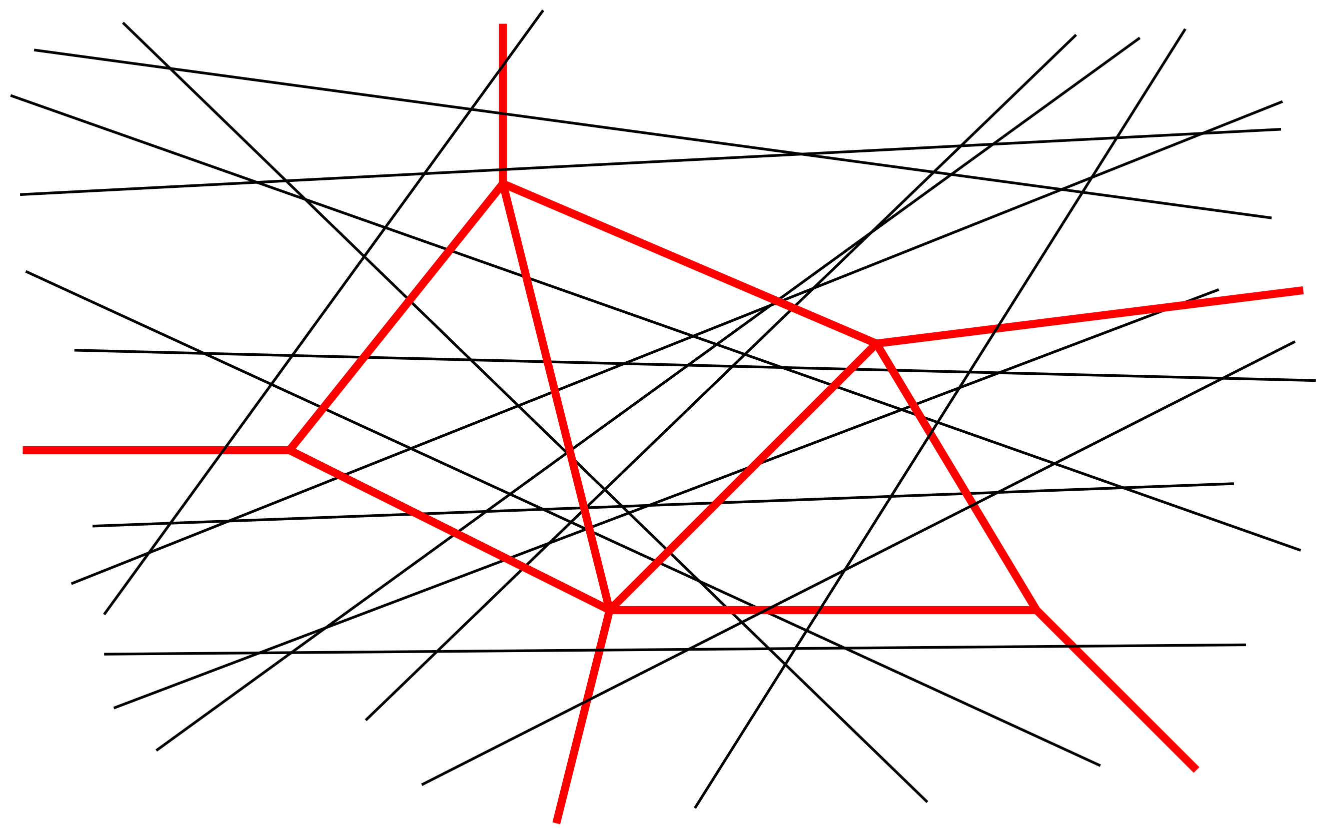
  over $u, v, u', v' \in [d]$

  (comparisons internal to $A_i + B_j$ are resolved if we know the location of A_i's point w.r.t. B_j's hyperplanes...)

Cutting Lemma

[Clarkson–Shor'89, Chazelle–Friedman'90]

- Given N hyperplanes in $\mathbb{R}^d$, can cut $\mathbb{R}^d$ into $d^{O(d)}r^d$ cells s.t. each cell intersects $O(N/r)$ hyperplanes



3SUM Alg'm 1 [C.'18]

- apply Cutting Lemma to $N = O(d^4 \cdot n/d)$ hyperplanes
- for each (i, j):
- Case 1: the hyperplanes of B_j do not intersect A_i's cell
  - all other A_i's in the same cell have "same" $A_i + B_j$
  - can pre-sort one $A_i + B_j$ per cell per B_j

$\Rightarrow$ total time $d^{O(d)} r^d \cdot n$

- Case 2: some hyperplane of B_j intersects A_i's cell
  - # such B_j's is $d^{O(1)} n/r$ per A_i
  - can pre-sort $A_i + B_j$ for each such A_i, B_j

$\Rightarrow$ total time $d^{O(1)} n/r \cdot n$

3SUM Alg'm 1 [C.'18]

- To search for for each $c \in C$:

  binary-search in $O(n/d)$ sorted lists of size d^2
  in $O((n/d) \log d)$ time

- total time $\widetilde{O}\left(d^{O(d)} r^d \cdot n \;+\; d^{O(1)} n/r \cdot n \;+\; n \cdot n/d\right)$

- set $r = d^{\Theta(1)} \;\Rightarrow\; \widetilde{O}(d^{O(d)} n + n^2/d)$ time

- set $d \approx \log n / \log\log n \;\Rightarrow\; \boxed{O^*(n^2/\log n)}$ time

3SUM Alg'm 2 [C.'18]

- Key Idea: Fredman's trick $+$ bit packing tricks

- Lemma 1: can do a batch of Q internal comparisons, in $\widetilde{O}(dn + Q/w)$ time on w-bit word RAM

  (Proof: to compare $A_i[u] + B_j[v]$ with $A_i[u'] + B_j[v']$, compare rank($A_i[u] - A_i[u']$) with rank($B_j[v'] - B_j[v]$)...)

- Lemma 2: can do a batch of Q internal selection queries, in $\widetilde{O}(dn + dQ/w)$ time on w-bit word RAM

  (Proof: simulate parallel alg'm for k-th smallest in $A_i + B_j$ with $O(\text{polylog}\, d)$ rounds of $O(d)$ comparisons, by Lemma 1...)

3SUM Alg'm 2 [C.'18]

- To search for each $c \in C$:

  binary-search in $O(n/d)$ lists of size d^2

  can simulate all binary searches with $O(\log d)$ calls to selection oracle from Lemma 2 with $Q = O(n \cdot n/d)$

- total time $\widetilde{O}(dn \; + \; n \cdot n/d \; + \; d \cdot n \cdot (n/d)/w)$

- set $d = \sqrt{n} \;\; \Rightarrow \;\; \widetilde{O}(n^{3/2} + n^2/w)$ time

- set $w \approx \log n \;\; \Rightarrow \;\; \boxed{O^*(n^2/\log n)}$ time again :-(

Final 3SUM Alg'm [C.'18]

Combine!

$$\Rightarrow \boxed{O^*(n^2/\log^2 n)} \text{ time}$$

Rest of Talk

1. Subquadratic Decision-Tree Upper Bounds

2. Slightly Subquadratic Alg'ms

3. Extensions to Other Problems

(median, +)-Convolution

similarly: $\boxed{O^*(n^2/\log^2 n)}$ time alg'm [C.'18]

“Algebraic” 3SUM

- Problem: Given sets A, B, C of n numbers, & fixed-degree algebraic function φ, decide $\exists a \in A$, $b \in B$, $c \in C$ with

$$\varphi(a, b) = c$$

- Fredman’s trick does not immediately extend!

- Barba–Cardinal–Iacono–Langerman–Ooms–Solomon [SoCG’17]: $\boxed{\widetilde{O}(n^{12/7})}$ comparisons for algebraic decision trees

“Algebraic” 3SUM

[Barba–Cardinal–Iacono–Langerman–Ooms–Solomon’17]

- New Idea: think geometrically. . . in 2D!

- map each group B_j into $O(d^2)$ curves:

  $$\{(x, x') : \varphi(x, B_j[v]) = \varphi(x', B_j[v'])\}$$

  over $v, v' \in [d]$

- map each group A_i into $O(d^2)$ points:

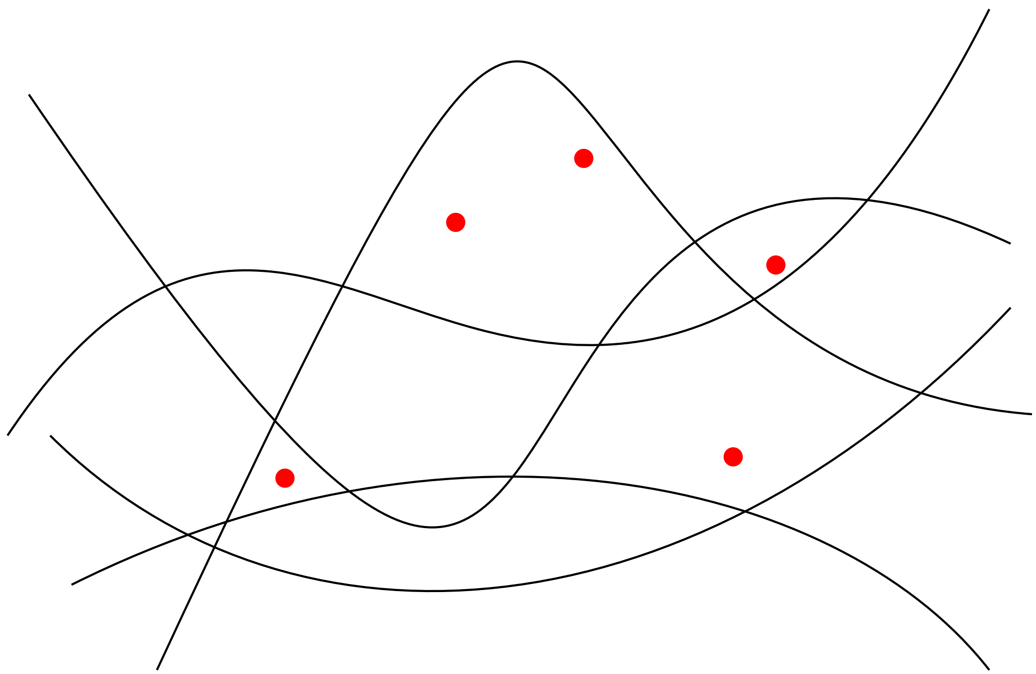
  $$(A_i[u], A_i[u']) \quad \text{over } u, u' \in [d]$$

  (comparisons internal to $\varphi(A_i, B_j)$ are resolved if we know the location of A_i’s points w.r.t. B_j’s curves. . .)

Range Searching Lemma

[Agarwal, Chazelle, Matoušek,. . . '80s/'90s]

- Given N points and N fixed-degree algebraic curves in 2D, can (implicitly) locate all the points w.r.t. the curves in $\widetilde{O}(N^{4/3})$ time



“Algebraic” 3SUM

[Barba–Cardinal–Iacono–Langerman–Ooms–Solomon’17]

- Preprocessing: apply Range Searching Lemma to $N = O(d^2 \cdot n/d)$ points & curves
  $\Rightarrow \widetilde{O}(N^{4/3}) = \widetilde{O}(dn)^{4/3}$ comps

- Rest: same!

- total # comps $\widetilde{O}((dn)^{4/3} + n \cdot n/d)$

- set $d = n^{2/7} \Rightarrow \boxed{\widetilde{O}(n^{12/7})}$ comps

- similarly: $\boxed{O^*(n^2/\log^2 n)}$ time alg’m [C.’18]

Algebraic 3SUM in s Dimensions

- Problem: Given sets A, B, C of n points in $\mathbb{R}^s$, & fixed-degree algebraic function φ, decide $\exists a \in A$, $b \in B$, $c \in C$ with

$$\varphi(a, b, c) = 0$$

- Application: 3COLLINEAR (for $s = 2$)...

- Range Searching Lemma generalizes, with $O(N^{2-1/O(s)})$ comps

- rest is same, except... grouping trick doesn't work :-(

Algebraic 3SUM in s Dimensions: Partial Results

- subquadratic upper bound for algebraic decision trees & $O^*(n^2/\log^2 n)$ time alg'm for
  - convolution version of algebraic 3SUM in const dimension s, e.g., "convolution 3COLLINEAR"
  - 3CONCURRENT for 3 sets of disjoint line segments in 2D
  - given 3 polygons in 2D, decide if their common intersection is empty

Main Open Questions

- 3COLLINEAR: $O(n^{2-\varepsilon})$ decision tree bound??

- 3SUM: $O(n^{2-\varepsilon})$ time??
  or better than $n^2/\log^2 n$ time?

  (($\mathrm{min}, +$)-convolution has $n^2/c^{\sqrt{\log n}}$ time alg'm by Williams [STOC'14] via the polynomial method...

  but it doesn't work for ($\mathrm{median}, +$)-convolution)

Integer 3SUM?

- general integer case:
  $O^*(n^2/\log^2 n)$ time by Baran–Demaine–Pătraşcu'05

- easy case, when input is in $[n^{2-\varepsilon}]$:
  $O(n^{2-\varepsilon})$ time by FFT

Integer 3SUM?

- clustered integer case, when input can be covered by $n^{1-\varepsilon}$ intervals of length n:
  $O(n^{2-\Omega(\varepsilon)})$ time by C.–Lewenstein [STOC'15]

  $\Rightarrow$ bounded monotone integer d-dimensional 3SUM in $O(n^{2-1/(d+O(1))})$ time

  $\Rightarrow$ jumbled string indexing over $[\sigma]$ in $O(n^{2-1/(\sigma+O(1))})$ time

  $\Rightarrow$ bounded-difference integer $(\min,+)$-convolution in $\widetilde{O}(n^{(9+\sqrt{177})/12}) = O(n^{1.86})$ rand. time

  – after preprocessing universe U of size n, can solve 3SUM for any subset of U in $\widetilde{O}(n^{13/7})$ time

Integer 3SUM?

- clustered integer case, when input can be covered by $n^{1-\varepsilon}$ intervals of length n:
  $O(n^{2-\Omega(\varepsilon)})$ time by C.–Lewenstein [STOC'15]

  (based on Balog–Szemerédi–Gowers (BSG) theorem from additive combinatorics: for any sets A, B, C of size N,

  if $|\{(a,b) \in A \times B : a+b \in C\}| = \Omega(\alpha N^2)$, then
  $\exists A' \subset A,\ B' \subset B$ both of size $\Omega(\alpha N)$ with
  $|A' + B'| = O((1/\alpha)^5 N))$